SCITECH
RESEARCH ORGANISATION|

Volume 7, Issue 1
March 03, 2018

Journal of Information Sciences and Computing Technologies
www.scitecresearch.com/journals

Unit Step and Impulse Function Equations to Simplify the Solution of Engineering Problems

Enrique Chicurel-Uziel
Instituto de Ingeniería, Universidad Nacional Autónoma de México,
Ciudad Universitaria, Coyoacán, 04510, Ciudad de México, México

Abstract

A unit step equation is proposed that when differentiated by elementary calculus yields the impulse function and when the resulting impulse function equation is integrated by elementary calculus yields the proposed unit step equation. Using these two equations, a two stage methodology is presented for the simplification of the solution of problems involving the impulse function.

Keywords: Riser unit step; two stage solution; impulse instant; post-impulse; elementary calculus differentiation; elementary calculus integration.

1. Introduction

There are a great number of publications referring to a vast diversity of mathematical representations of the impulse function, or Dirac delta function, each suited for a different application, Refs.[1 to 7] to mention only a few recent ones. Most of them are highly developed and rather complex. In this paper a simple delta representation is proposed suitable for most, if not all, engineering applications.

2. Motivation

Nomenclature:
H(t)= Heaviside (riserless) unit step
R(t)= unit step with a riser
t = time
δ(t) = impulse function, otherwise known as the Dirac delta
ϵ = a very short time

The Heaviside Unit Step
The Heaviside unit step is very often defined thus:

$$H(0)\begin{cases} = 0 & t < 0 \\ Undefined & t = 0 \\ = 1 & t > 0 \end{cases} \qquad (1)$$

According to elementary calculus, the derivative of a function at a given point is the value of its slope at that point, thus the derivative of the Heaviside unit step(1) should be:

$$\frac{dH(0)}{dt}\begin{cases} = 0 & t < 0 \\ Undefined & t = 0 \\ = 0 & t > 0 \end{cases} \qquad (2)$$

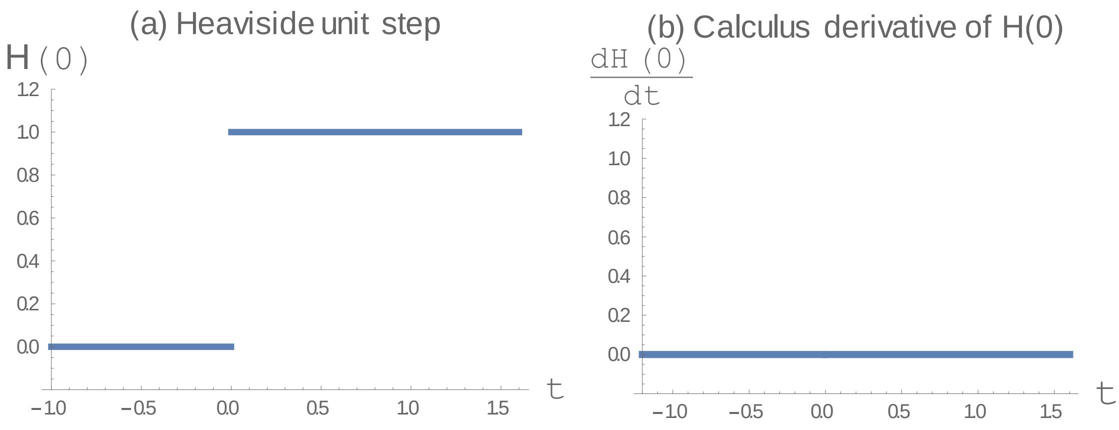


Fig 1: (a) Piecewise plot of the Heaviside unit step function. Notice that there is no riser (vertical line) at the "jump", it is empty. (b) Elementary calculus derivative of the Heaviside unit step function obtained by applying the derivative function of *Mathematica* to the piecewise function H(0). See Appendix A.

This is confirmed by applying the derivative operator of *Mathematica* to the piecewise function (1), Fig. 1(a). However, the vast majority of textbooks, of pure and applied mathematics, physics and even of engineering consider the derivative of the Heaviside unit step (1) to be:

$$\frac{dH(0)}{dt}\left\{\begin{matrix} = 0 & t < 0 \\ = \infty & t = 0 \\ = 0 & t > 0 \end{matrix}\right\} = \delta(0) \tag{3}$$

How can there be a slope and a derivative at the jump point, *t=0*, of the step function (1) where the function is not defined? The answer to this question is that δ (the Dirac delta or the impulse function) is not a function, it is a distribution and it does have a distributional derivative. But engineers and many other users of the delta, interested in its applications, are not acquainted with the Theory of Distributions [8, 9].

Furthermore, the textbooks obtain solutions of problems, represented by a differential equation involving the impulse function, using the principles of calculus to handle the equation and using the theory of distributions to represent the impulse function in spite of the disagreement between relations (2) and (3). This is disconcerting for the author and he imagines that it must be disconcerting for many others. A remedy was proposed for this situation in Refs.[10-12]: fill the jump of the unit step with a vertical line segment, i.e., a riser. But verticality required two parametric equations to represent the unit step.

In this paper non-parametric piecewise equations are presented of both the unit step and the impulse function such that an elementary calculus (non-distributional)differentiation of the unit step equation yields the impulse function equation and an elementary calculus integration of the impulse function equation yields the unit step equation. These equations areas close to the exact representations as the user chooses them to be. Furthermore, in this case only one equation is required to represent each of these relations.

Referring to problems that involve the impulse function, in this paper, instead of the usual single governing equation, two equations are proposed: the first equation to represent the process that takes place during the impulse instant and the second to represent the process that takes place during post-impulse time. Each of these two equations is easier to solve than the usual single equation. Unlike the conventional solution, the solution of the proposed equations comply with the real initial condition.

3. Proposed representations of the unit step and the Dirac delta

Unit step with a riser

The proposed unit step with a riser is shown in Fig. 2, it is made up of three line segments:

$$R(0)\left\{\begin{matrix} = 0 & t \le 0 \\ = \frac{t}{\epsilon} & 0 \le t \le \epsilon \\ = 1 & t \ge \epsilon \end{matrix}\right. \tag{4}$$

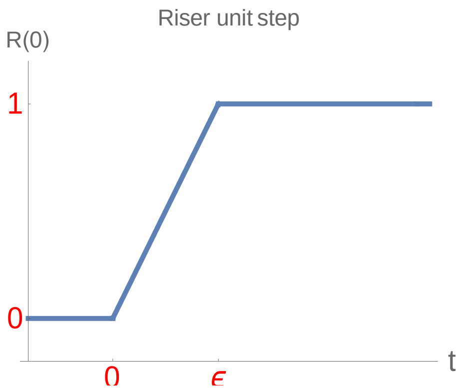


Fig 2: Plot of the proposed unit step equation, with a riser with a very high value of ϵ to illustrate the concept of the representation.

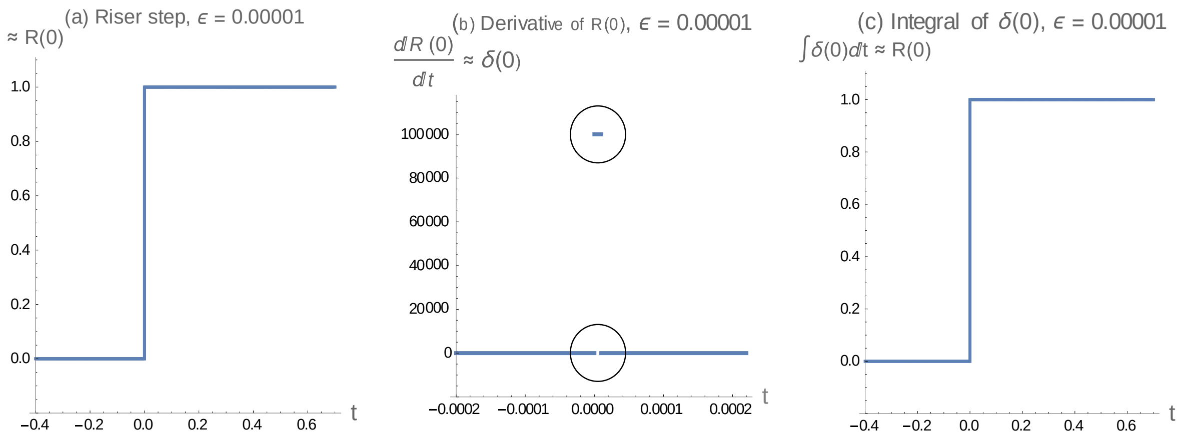


Fig 3: (a) Unit step with a riser, R(0). (b) Applying the *Mathematica* differentiation operator to R(0) yields the rectangular approximation of the Dirac delta, $\delta(0)$. (c) Applying the *Mathematica* integration operator to $\delta(0)$ yields R(0). See Appendix B.

The piecewise function (4) is an approximation but it is as close to the exact function as it is desired to make it by simply choosing an ϵ as small as required for the particular application. Fig. 2 was plotted using $\epsilon = 0.34$ which is a very high value of ϵ in order to make clear the idea behind this representation. Figs. 3a, 3b and 3c were plotted with $\epsilon = 0.00001$ which is a more likely value to be used in the solution of a meaningful problem.

The derivative of function (4) is simply:

$$\frac{dR(0)}{dt}\left\{\begin{array}{cc} =0 & t<0 \\ =\frac{1}{\epsilon} & 0\le t<\epsilon \\ =0 & t\ge\epsilon \end{array}\right\}\approx\delta(0) \qquad (5)$$

The piecewise function (5) is what is known as the rectangular approximation of the Dirac delta and, just as in the case of function (4), it is as good an approximation as ϵ is small. This is satisfactory for most engineering applications.

4. Adequacy of the representation

Compliance with the definition

Values

The proposed function (5) has the appropiate values, as far as engineering applications are concerned:

Area

The area under the proposed function is

$$A = \int_{-\infty}^{+\infty} \delta(0)dt$$

$$A = \int_{-\infty}^{0} (0)dt + \int_{0}^{\epsilon} \frac{1}{\epsilon}dt + \int_{\epsilon}^{+\infty} (0)dt$$

$$A = 0 + \frac{1}{\epsilon}[t]_0^\epsilon + 0$$

$\therefore \ A = 1$ (6)

It is clear that the proposed function has the same area as the Dirac delta.

Fundamental or sifting property

$f(t) =$ a continous function of t

$$P = \int_{-\infty}^{+\infty} \delta(0)f(t)dt$$

or in accordance with Eq. (5):

$$P = \int_{-\infty}^{0} (0)f(t)dt + \int_{0}^{\epsilon} \frac{1}{\epsilon}f(t)dt + \int_{\epsilon}^{+\infty} (0)f(t)dt$$

$$P = 0 + \frac{1}{\epsilon}\int_{0}^{\epsilon} f(t)dt + 0$$

For a small enough ϵ:

$$\frac{1}{\epsilon}\int_{0}^{\epsilon} f(t)dt = \frac{1}{\epsilon}\int_{0}^{\epsilon} f(0)dt = \frac{1}{\epsilon}f(0)\int_0^\epsilon dt = \frac{1}{\epsilon}f(0)\epsilon = f(0)$$

$\therefore \ P = f(0)$

$\therefore \int_{-\infty}^{+\infty} \delta(0)f(t)dt = f(0)$ (7)

This last equation is the fundamental property, so it has been proven that the proposed function complies with it.

5. Examples

Example 1. Heat transfer problem. Linear system with 2 independent variables.

Consider a one dimensional rod subject to an impulsive heat source with initial temperature of 0°C along its full length and with the ends kept at 0°C throughout the whole process. Determine the temperature T as a function of the time t and the position along the length of the rod x.

Nomenclature
c = specific heat

L = length of rod
q = heat energy
Q = q/V
m = mass
t = time
T(x,t) = temperature
V = volume
x = position along the rod
ρ = m/V

The governing equation is:

$$c\rho\frac{\partial T}{\partial t}-k\frac{\partial^2 T}{\partial x^2}=Q\delta(0)\,. \tag{8}$$

(Here *Q* has units of *energy/volume*)

Subject to the boundary conditions:

$$\begin{aligned}T(0,t)&=0\\T(L,t)&=0\end{aligned} \tag{9}$$

and to the initial condition:

$$T(x,0)=0\,. \tag{10}$$

Solution

Unlike the conventional solution, ***two separate periods*** will be considered: the ***impulse instant*** and the ***post-impulse time***.

$$c\rho\frac{\partial T}{\partial t}-k\frac{\partial^2 T}{\partial x^2}=Q\delta(0) \tag{11}$$

Impulse instant

During the impulse instant, $0\leq t\leq\epsilon$, due to the sudden application of heat there is an immediate increase in temperature so both $Q\delta(0)$ and $c\rho\frac{\partial T}{\partial t}$ are most significant. Since this process takes place during the very short time ϵ(ideally zero time) there is no time for conduction, represented by $k\frac{\partial^2 T}{\partial x^2}$, to take place. Thus it is clear that, during the impulse instant, $k\frac{\partial^2 T}{\partial x^2}$ is negligible so that the governing equation for this period is:

$$c\rho\frac{dT}{dt}=Q\delta(0) \tag{12}$$

Substituting Eq. (5):

$$c\rho\frac{dT}{dt}=Q\frac{dR(0)}{dt}$$

which reduces to:

$$c\rho dT=QdR \tag{13}$$

Integrating:

$$c\rho T = QR + C.$$

Substituting the value of R for the impulse instant given in Eq.(4):

$$c\rho T = Q\frac{t}{\epsilon} + C$$

at $t = 0 \qquad T = 0 \qquad \therefore C = 0$

$$T = \frac{Q}{c\rho}\frac{t}{\epsilon} \tag{14}$$

Eq. (14) represents the process that takes place during the impulse instant.

At the end of the impulse instant $t = \epsilon$, substituting this value in Eq.(14) yields:

$$T(x, \epsilon) = \frac{Q}{c\rho} \tag{15}$$

Post-impulse

At post-impulse time, $t \geq \epsilon$, according to Eq. (5) $\delta(t, 0) = 0$ and the problem to be solved now is the ***homogeneous*** equation:

$$c\rho\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \tag{16}$$

subject to the boundary conditions:

$$\begin{aligned} T(0,t) &= 0 \\ T(L,t) &= 0 \end{aligned} \tag{17}$$

and to the initial time condition (the same as the final time of the impulse instant):

$$T(x, \epsilon) = \frac{Q}{\rho c} \tag{18}$$

Eq.(16) is ***separable*** and the problem made up of Eqs. (16)-(18) has the well known solution (see, for instance, Ref. [13]):

$$T_p = \frac{Q}{\rho c}\sum_{n=1}^{\infty}\frac{2(1-\cos n\pi)}{n\pi} e^{\frac{n^2\pi^2}{L^2}\frac{k}{\rho c}t} \sin\left(\frac{n\pi}{L}x\right) \tag{19}$$

therefore, the complete solution in dimensionless form is:

$$\left(\frac{\rho c}{Q}T\right) = \begin{cases} = 0 & t < 0 \\ = \frac{t}{\epsilon} & 0 \leq t < \epsilon \\ = \sum_{n=1}^{\infty}\frac{2(1-\cos n\pi)}{n\pi} e^{n^2\pi^2\left(\frac{k}{\rho c L^2}t\right)} \sin\left[n\pi\left(\frac{x}{L}\right)\right] & \epsilon \leq t \end{cases} \tag{20}$$

The dimensionless variables are the ones enclosed in the round and tall parenthesis.

Figure 4 is a plot of the conventional solution, equation (19). Figure 5 is a plot of equations (20).

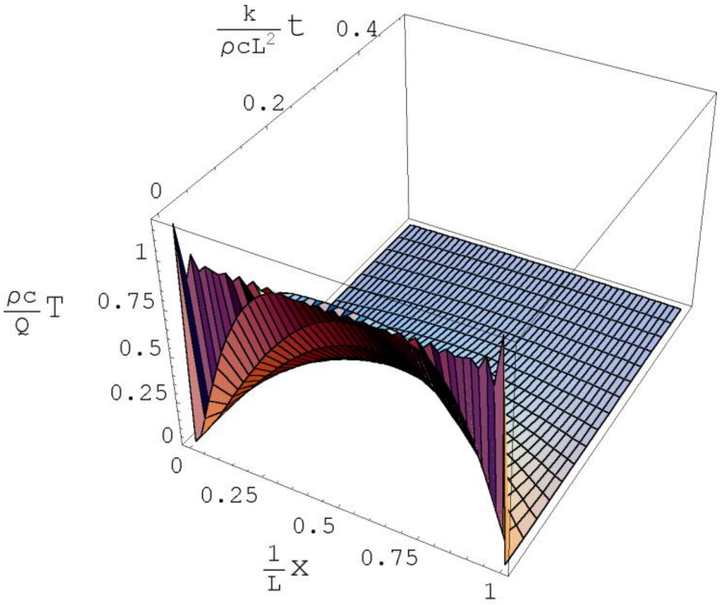


Fig 4: Plot of the conventional solution, equation (19). Notice that the initial temperature seems to be: $\frac{\rho c}{Q} T(x,0) = 1$, while the specified condition is $\frac{\rho c}{Q} T(x,0) = 0$. The oscillations of the pseudo initial temperature along the full length of the rod are spurious, they are due to the Gibbs phenomenon.

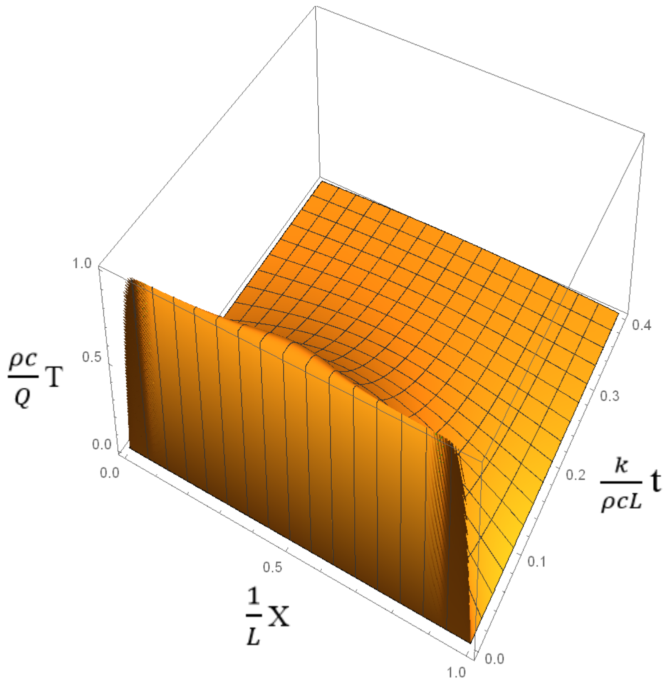


Fig 5: Plot of equations (20) using a value of ϵ = 0.000000000001. The present solution clearly shows the initial condition to be zero, i.e., $\frac{\rho c}{Q} T(x,0) = 0$, *strictly in accordance with the specified condition, Eq. (10)*, and, furthermore it shows the initial, *instantaneous process* of the change of temperature, i.e., $\Delta(\frac{\rho c}{Q} T) = 1$.

Example 2. Mechanics problem. Nonlinear system.

The response of a pendulum subjected to an initial impulse of magnitude I will be obtained in the form of the velocity, *v*, as a function of the height, *y*, of the oscillating mass, i.e., the phase-plane solution.

Nomenclature
g = acceleration due to gravity
I = impulse magnitude
L = length of pendulum arm
m = mass of pendulum
s(t) = θL
v(t) = s/t = velocity of pendulum mass
x(t) = horizontal position of pendulum mass
y(t) = vertical position (height) of pendulum mass
θ(t) = angular position of pendulum mass
ω(t) = angular velocity of pendulum mass

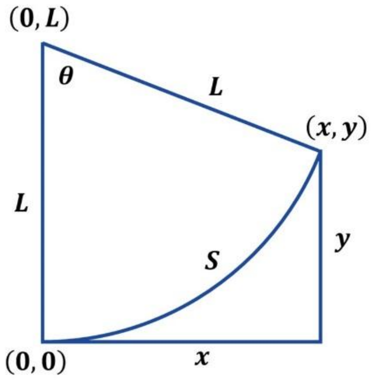


Fig 6: Oscillating pendulum.

Referring to Fig.6, the governing equation is:

$$mL\frac{d^2\theta}{dt^2} + mg\ sin\theta = I\delta(t,0) \tag{21}$$

and the initial conditions are:

$$\theta(0) = 0 \tag{22}$$

$$\omega(0) = 0 \tag{23}$$

Or equivalently:

$$m\frac{dv}{dt} + mg\ sin\theta = I\delta(t,0) \tag{24}$$

$$s(0) = 0 \tag{25}$$

$$v(0) = 0 \tag{26}$$

Solution

Impulse Instant, $0 \le t \le \epsilon$

During the impulse instant the excitation,$I\delta(t,0)$, is, of course, very high and this induces an instantaneous sharp increase in $m\frac{dv}{dt}$. Furthermore, there is no time for any motion to take place, thus $mg\ sin\theta$ may be neglected so that, during the impulse instant, the governing equation is:

$$m\frac{dv}{dt} = I\delta(t,0) \tag{27}$$

and, of course, the initial conditions are given by Eqs. (25) and (26).

Substituting Eq. (5) into Eq.(27):

$$m\ \frac{dv}{dt} = I\ \frac{dR(t,0)}{dt} \tag{28}$$

or, equivalently:

$$m\ dv = I\ dR \tag{29}$$

Integrating:

$$m\int_{v(0)}^{v(t)} dv = I\int_{R(0)}^{R(t)} dR \tag{30}$$

From Eqs.(26) and (4).: $v(0) = 0,\ R(0) = 0,\ R(t) = \frac{t}{\epsilon}$. Substituting these values into Eq. (30), integrating and solving for $v(t)$:

$$v(t) = \frac{I}{m}\frac{t}{\epsilon} \tag{31}$$

Eq. (31) represents the process that takes place during the impulse instant.

At the end of the impulse instant, $t = \epsilon$ and $v = v(\epsilon)$. Substituting these values into Eq. (31) yields the value of the velocity at that time:

$$v(\epsilon) = \frac{I}{m}\omega(\epsilon) = \frac{I}{mL} \tag{32}$$

Post Impulse, $t \geq \epsilon$

In accordance with Eqs. (5) and (25), in this period, the governing equation is:

$$L\frac{d^2\theta}{dt^2} + g\ sin\theta = 0 \tag{33}$$

or equivalently:

$$L\omega\frac{d\omega}{d\theta} + gsin\theta = 0 \tag{34}$$

$$\int_{\omega(\epsilon)}^{\omega} \omega d\omega = -\frac{g}{L}\int_{0}^{\theta} sin\theta d\theta \tag{35}$$

integrating and solving for ω:

$$\omega = \pm\sqrt{\frac{2g}{L}(cos\theta - 1) + \omega(\epsilon)^2} \tag{36}$$

or:

$$v = \pm\sqrt{2gL(cos\theta - 1) + v(\epsilon)^2} \tag{37}$$

According to Fig. 6:

$$y = L(1 - cos\theta) \tag{38}$$

Substituting the first of Eqs.(32) and Eq.(38) into Eq.(37) yields:

$$v = \pm\sqrt{\frac{I^2}{m^2} - 2gy} \tag{39}$$

or in dimensionless form:

$$\left(\frac{m}{I}v\right) = \pm\sqrt{1 - 2\left(\frac{m^2 g}{I^2}y\right)} \tag{40}$$

Using Eqs.(31) and (40) the following two dimentionless equations are obtained:

$$\frac{m}{I}v\begin{cases} = \frac{y}{\epsilon} & 0 \le y < \epsilon \\ = +\sqrt{1 - 2\frac{m^2 g}{I^2}y} & y \ge \epsilon \end{cases} \tag{41}$$

$$\frac{m}{I}v = -\sqrt{1 - 2\frac{m^2 g}{I^2}y} \qquad y \ge \epsilon \tag{42}$$

The plot of Fig.7 was obtained by superposing the plots of Eqs.(41) and (42).

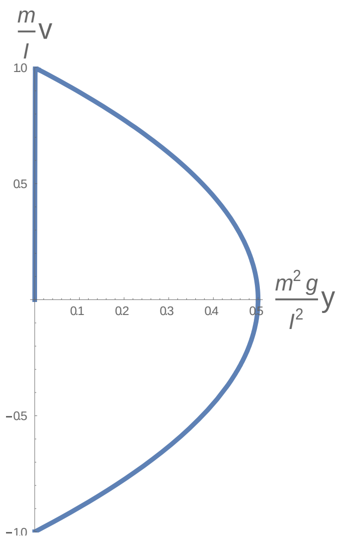


Fig 7: Plot of the phase-plane solution.

6. Conclusions

An equation for the unit step has been presented in this paper which when differentiated by ***elementary calculus*** yields the equation of the Dirac delta.Integrating the resulting Dirac Delta by ***elementary calculus*** yields the equation of the unit step. Both of these equations are approximate, but they are as close to the exact values as the user choses them to be.

Also with reference to a problem involving the impulse function, a methodology has been proposed whereby, instead of the usual single governing equation there are two: the impulse instant equation and the post-impulse equation. The proposed equations are, most often, easier to solve than the usual equation and they yield a more complete solution. The solution of the impulse instant equation complies with the true initial conditions and, furthermore, it yields the initial conditions of the post impulse. The impulse instant equation is perfectly reconciled with our intuition.

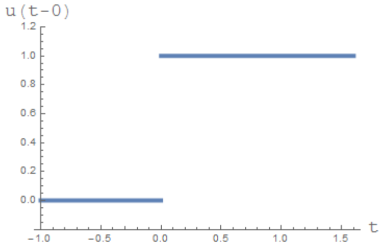
Since there is no need to resort to the Theory of Distributions, the method presented here may be taught at an earlier stage.

Acknowledgements

I wish to acknowledge the support of the Instituto de Ingeniería UNAM. I am grateful to Fo-Lyn Yong Primero for her aid in generating the plots. I am very grateful to Filiberto Gutiérrez for his most valuable assistance.

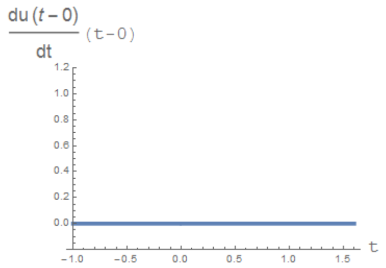
Appendix A. *Mathematica* Differentiation of the Heaviside unit step.

```
u[t_, a_] := Piecewise[{{0, t < a}, {1, t > a}}]
            función a trozos

Plot[u[t, 0], {t, -1, 1.6}, PlotRange → {-0.2, 1.2}, AxesOrigin → {-1, -0.2}, PlotStyle → {{Thickness[0.013]}}, AxesLabel → {"t", "u(t-0)"}]
representación gráfica    rango de representación    origen de ejes    estilo de represen··· grosor    etiqueta de ejes
```



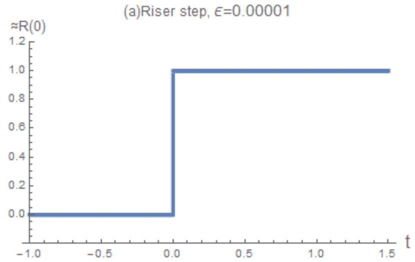

```
du = D[u[t, 0], t]
     deriva

{ 0              t < 0 || t > 0
  Indeterminate  True
```

```
Plot[du, {t, -1, 1.6}, PlotRange → {-0.2, 1.2}, AxesOrigin → {-1, -0.2}, PlotStyle → {{Thickness[0.013]}}, AxesLabel → {"t", "du(t-0)/dt (t-0)"}]
representación gráfica    rango de representación    origen de ejes    estilo de represen··· grosor    etiqueta de ejes
```



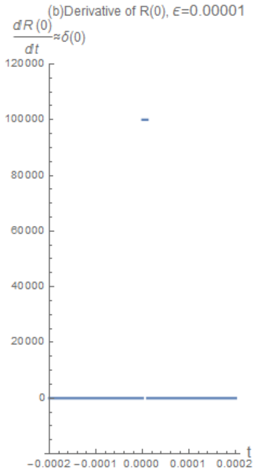
Appendix B. *Mathematica* differentiation of R(0). *Mathematica* integration of δ(0).

```
e = 0.00001

stp = Piecewise[{{0, -1 < t < 0}, {t/e, 0 < t < e}, {1, e < t < 2}}]
      función a trozos

{ 0          -1 < t < 0
  100000. t  0 < t < 0.00001
  1          0.00001 < t < 2
  0          True

g1 = Plot[stp, {t, -1, 1.5}, PlotStyle → {Thickness[0.01]}, PlotPoints → 140000, AspectRatio → Automatic, AxesOrigin → {-1, -0.2},
     representación gráfica   estilo de represe··· grosor   número de puntos en la repre···   cociente de aspecto automático   origen de ejes
  PlotRange → {-0.2, 1.2}, PlotLabel → "(a)Riser step, ϵ=0.00001", AxesLabel → {"t", "≈R(0)"}]
  rango de representación   etiqueta de representación   etiqueta de ejes
```

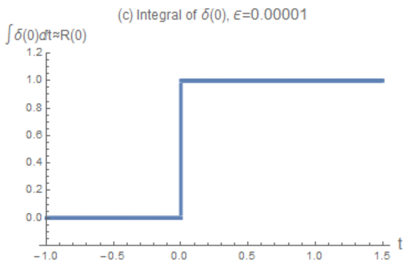



```
del = D[stp, t]
      deriva

{ 0              t < 0
  100000.        0 < t < 1/100000
  0              1/100000 < t < 2 || t > 2
  Indeterminate  True
```

```
g2 = Plot[del, {t, -0.0002, 0.0002}, PlotStyle → {Thickness[0.013]}, AspectRatio → 2, PlotPoints → 500, AxesOrigin → {-0.0002, -20000},
  PlotRange → {-20000, 120000}, PlotLabel → "   (b)Derivative of R(0), ϵ=0.00001", AxesLabel → {"t", "dR(0)/dt ≈δ(0)"}]
```




```
idel = Integrate[del, t]

  0.          t ≤ 0.
{ 100000. t   0. < t ≤ 0.00001
  1.          True

g3 = Plot[idel, {t, -1, 1.5}, PlotStyle → {Thickness[0.01]}, PlotPoints → 130000, AspectRatio → Automatic, AxesOrigin → {-1, -0.2},
  PlotRange → {-0.2, 1.2}, PlotLabel → "(c) Integral of δ(0), ϵ=0.00001", AxesLabel → {"t", "∫δ(0)dt≈R(0)"}]
```



References

[1] B. Engquist, A. K. Tornberg and R. Tsai, *Discretization of Dirac delta functions in level set methods*, Journal of Computational Physics archive, Vol. 207 Issue 1, July(2005), pages 28-51.

[2] X. Xu, *Nonlinear trigonometric approximation and the Dirac delta function*, Journal of Computational and Applied Mathematics 209 (2006) 234 – 245.

[3] P. Smereka, *The numerical approximation of a delta function with application to level set methods*. Journal of Computational Physics 211 (2006) 77–90.

[4] C. Min and F. Gibou, *Robust second-order accurate discretizations of the multi-dimensional Heaviside and Dirac delta functions*, Journal of Computational Physics archive, Vol. 227, Issue 22, Nov.(2008), pages 9686-9695.

[5] M. Jauregui and C. Tsallis, *New representations of π and Dirac delta using the nonextensive-statistical-mechanics q-exponential function*, Journal of Mathematical Physics 51, 063304,(2010).

[6] A. Chevreuil, A. Plastino, and C. Vignat, *On a conjecture about Dirac´s delta representation using q-exponentials*, Journal of Mathematical Physics 51, 093502,(2010).

[7] Y.T. Li and R. Wong, *Integral and Series Representations of the Dirac Delta Function*, Communications on Pure and Applied Analysis,(2013). arXiv:1303.1943[math.CA].

[8] L. Schwartz, *Théorie des distributions*, Paris, Hermann, (1966).

[9] M. A. Al-Gwaiz, *Theory of distributions*, CRC Press, (2012).

[10] E. Chicurel-Uziel, *Dirac delta representation by exact parametric equations. Application to impulsive vibration systems*, J. of Sound and Vibration, Vol. 305,134, (2007).

[11] E. Chicurel-Uziel and F. A. Godínez-Rojano, (2013) *Parametric Dirac delta to simplify the solution of linear and nonlinear problems with an impulsive forcing function,* Journal of Applied Mathematics and Physics, Vol. 1, 16-25. http://dx.doi.org/10.4236/jamp.2013.17003

[12] E. Chicurel-Uziel and F. A. Godínez-Rojano, (2015) *Parametrization to improve the Solution Accuracy of Problems Involving the Bi-Dimensional Dirac Delta in the Forcing Function*, Journal of Applied Mathematics and Physics, **3**,1168-1177. http://dx.doi.org/10.4236/jamp.2015.39144

[13] R. Haberman, *Applied Partial Differential Equations*, 4th Ed., Pearson, Prentice Hall, Upper Saddle River, New Jersey, US,(2004), pages 51-53.