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Factorization approach to generalized Dirac oscillators

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Abstract

We study generalized Dirac oscillators with complex interactions in (1+1) dimensions. It is shown that for the choice of interactions considered here, the Dirac Hamiltonians are η pseudo Hermitian with respect to certain metric operators η. Exact solutions of the generalized Dirac Oscillator for some choices of the interactions have also been obtained. It is also shown that generalized Dirac oscillators can be identified with Anti Jaynes Cummings type model and by spin flip it can also be identified with Jaynes Cummings type model.

Keywords: Dirac oscillators; pseudo Hermitian; Factorization Method; Partner Potential.

1 Introduction

The Dirac oscillator which is linear in both momenta and coordinates is one of a few relativistic systems admitting exact solutions [1-3]. This system has many applications and over the years it has been studied extensively by a number of authors [4]. Various exactly solvable generalizations of the Dirac oscillator have also been proposed [5]. On the other hand during the last decade non Hermitian interaction in non relativistic [6] as well as relativistic quantum mechanics [7] have been examined from various points of view. One of the main interest in the study of such systems is that a class of potentials, namely the PT symmetric [8] and ç-pseudo Hermitian [9] ones admit real eigenvalues despite being non Hermitian. Analogues of some of these non Hermitian systems have been found in optics [10] and have also been realized experimentally [11]. It may be noted that relativistic non Hermitian (PT symmetric) interactions can be realized in optical structures [12]. Also there exists photonic realization of the (1 + 1) dimensional Dirac oscillator [13]. Here we shall consider η-pseudo Hermitian interactions in the context of relativistic quantum mechanics [14, 15]. To be more specific, we shall examine η-pseudo Hermitian interactions within the framework of generalized Dirac oscillator in (1 + 1) dimensions. In particular, we shall obtain a class of interactions which are η-pseudo Hermitian and the metric operator η will also be found explicitly. Subsequently we shall employ the mapping between the Dirac oscillator and the Jaynes Cummings (JC) model [16–18] to obtain a class of exactly solvable non Hermitian JC as well as anti Jaynes Cummings (AJC) type models. The rest of the paper is organized as follows: sec. II introduces the generalized Dirac oscillator system and discusses the conditions for which it is pseudo hermitian. Sec. III provides two explicit examples. Sec. IV discusses the relation with the generalized AJC and JC type models while sec. V contains the conclusions.

2 Generalized Dirac equation

As we know the Hamiltonian of the Dirac oscillator in $(1+1)$ dimensions is given by [19],

$$H = c\sigma_x(p_x - i\beta\omega x) + \beta mc^2 \tag{1}$$

where c is the velocity of light and σ_x and $\beta = \sigma_2$ are the standard pauli matrices given by,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2}$$

Here we note that the generalization of this system can be constructed by replacement $m\omega x \to v(x) \to g(x)$ so, the corresponding Hamiltonian (1) will be as,

$$H = \begin{bmatrix} mc^2 & cp_x + icg(x) \\ cp_x - icg(x) & -mc^2 \end{bmatrix}, \tag{3}$$

an the corresponding eigenvalue equation reads,

$$H\Psi = E \quad \Psi, \Psi = \binom{\Psi_1}{\Psi_2}, \tag{4}$$

In terms of the components the above equation reads,

$$[p_x + ig(x)]\Psi_2 = (\frac{E - mc^2}{c})\Psi_1 \tag{5}$$

$$[p_x - ig(x)]\Psi_2 = (\frac{E + mc^2}{c})\Psi_1.$$

So the corresponding Schrödinger equation will be as,

$$(-\hbar^2 \frac{d^2}{dx^2} + V_-(x))\Psi_1 = \varepsilon\Psi_1 \tag{6}$$

$$(-\hbar^2 \frac{d^2}{dx^2} + V_+(x))\Psi_2 = \varepsilon\Psi_2,$$

where $V_\pm(x) = g^2(x) + \hbar g'(x)$ and $\varepsilon = \dfrac{E^2 - m^2c^4}{c^2}$.

3 Example of generalized Dirac equation and factorization method

Now we consider two different examples of pseudo Hermitian interactions by choosing suitable expressions for $g(x)$. Also we that a Hamiltonian H is said to be η pseudo Hermitian if it satisfies the relation [–9] $H^+ = \eta H \eta^{-1}$. We also note that the form of η will be as,

$$\eta = e^{-\theta P_x}, \tag{7}$$

where θ is a real parameter. Then it can be shown that η has the following properties [20],

$$\eta c \eta^{-1} = c, \eta P_x \eta^{-1} = P_x, \eta V(x) \eta^{-1} = V(x + i\hbar\theta),$$

$$\eta\Psi(x)\eta^{-1} = \Psi(x + i\hbar\theta). \tag{8}$$

In order to obtain exact solution of the problem we use the $g(x) = D - (A + iB)e^{-\alpha x}$ in equation (6), so we have,

$$(-\hbar^2 \frac{d^2}{dx^2} + D^2 - (A + iB)^2 e^{-2\alpha x} - (2D + \hbar\alpha)(A + iB)e^{-\alpha x})\Psi_1 = \varepsilon_n \Psi_1, \tag{9}$$

with comparing by following Laguree and,

$$(-\hbar^2 \frac{d^2}{dx^2} + D^2 - (A + iB)^2 e^{-2\alpha' x} - (2D + \hbar\alpha')(A + iB)e^{-\alpha' x})\Psi_1 = \varepsilon_n \Psi_1, \tag{10}$$

Now we take following variable,

$$\alpha = \frac{D}{\hbar\alpha'} - n + 1, \tag{11}$$

$$z = 2\frac{A+iB}{\hbar\alpha'} e^{-\alpha' x},$$

and choose $\Psi_1 = NF(z)L(\alpha,\beta)_{n,m}(z)$, where $F(z) = z^{-1}e^{-\frac{z}{2}}$, where $\alpha = \frac{D}{\hbar\alpha} - n + 1$, and $n = 0,1,2,...$, $m = 0$, and $\beta = 1$. Here we use the Normalization condition as,

$$\int \Psi_1(z)\Psi_2^*(z)dz = 1. \tag{12}$$

In order to obtain eigenfunction and eigenvalue for the generalized Dirac equation we compare equation (10) with the following associated Laguerre differential equation [17,18,19],

$$zL_{n,m}^{''(\alpha,\beta)}(z) + (1+\alpha-\beta z)L_{n,m}^{'(\alpha,\beta)}(z) +$$

$$\left[(n-\frac{m}{2})\beta - \frac{m}{2}(\alpha+\frac{m}{2})\frac{1}{z}\right]L_{n,m}^{(\alpha,\beta)}(z) = 0, \tag{13}$$

where for our case we have $m = 0$, $\beta = 1$. So, the corresponding wave function and energy spectrum will be as,

$$\Psi_1 = Nz^{\frac{\alpha-1}{2}} e^{-\frac{z}{2}} L_n^{(\alpha,1)}(z), \tag{14}$$

and,

$$\varepsilon_n = [D^2 - (D - n\hbar\alpha')^2]. \tag{15}$$

So, the solution associated Laguerre in the Rodrigues representation are,

$$L_n^{(\alpha,1)}(z) = \frac{a_n(\alpha,\beta)}{z^\alpha e^{-z}}(\frac{d}{dz})^n (z^{n+\alpha}e^{-z}) \tag{16}$$

where,

$$a_n(\alpha,\beta) = \sqrt{\frac{1}{\Gamma(n+1)\Gamma(n+\alpha+1)}} = \frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}} = \frac{1}{\sqrt{n!(n+\alpha)!}}. \tag{17}$$

In here we also discuss the raising and lowering operator which is corresponding to the generalized Dirac equation. So, we can factorize the associated Laguerre differential equation for the corresponding problem with respect to the parameter n as follows,

$$A_n^+(z)A_n^-(z)L_n^{(\alpha,1)}(z) = n(n+\alpha)L_n^{(\alpha,1)}(z), \tag{18}$$

$$A_n^-(z)A_n^+(z)L_{n-1}^{(\alpha,1)}(z) = n(n+\alpha)L_{n-1}^{(\alpha,1)}(z). \tag{19}$$

So, the differential operators as functions of parameters n will be as,

$$A_n^+(z) = z\frac{d}{dz} - z + n + \alpha,$$

$$A_n^-(z) = -z\frac{d}{dz} + n \tag{20}$$

Note that the shape invariance equation (18),(19) can also be written and the raising and lowering relation,

$$A_n^+(z)L_{n-1}^{(\alpha,1)}(z)=\sqrt{n(n+\alpha)}L_n^{(\alpha,1)}(z),$$

$$A_n^-(z)L_n^{(\alpha,1)}(z)=\sqrt{n(n+\alpha)}L_{n-1}^{(\alpha,1)}(z). \tag{21}$$

Also, we note the Ψ can be written as $\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}$, so we must to sole also Ψ_2 and obtain the ε_n and corresponding wave function. As a before the generalized Dirac equation for the Ψ_2 will be as,

$$(-\hbar^2\frac{d^2}{dx^2}+D^2+(A+iB)^2e^{-2\alpha' x}-(2D-\hbar\alpha')(A+iB)e^{-\alpha' x})\Psi_2=\varepsilon_n\Psi_2, \tag{22}$$

where,

$$\alpha=\frac{D}{\hbar\alpha'}-n+1, \tag{23}$$

$$z=2\frac{A+iB}{\hbar\alpha'}e^{-\alpha' x}.$$

Again we take $\Psi_2=F(z)L(\alpha-1,\beta)_{n,m}(z)$, where $F(z)=Nz^{\alpha-2}e^{-\frac{z}{2}}$, and $n=0,1,2,...$, $m=0$, and $\beta=1$. By comparing the associate Laguerre equation we obtain ε_n as,

$$\varepsilon_n=[D^2-(D-(n+1)\hbar\alpha')^2]. \tag{24}$$

So, the solution associated Laguerre in the Rodrigues representation are,

$$L_n^{(\alpha-1,1)}(z)=\frac{a_n(\alpha-1,\beta)}{z^{\alpha-1}e^{-z}}(\frac{d}{dz})^n(z^{n+\alpha-1}e^{-z}) \tag{25}$$

where,

$$a_n(\alpha-1,\beta)=\frac{1}{\sqrt{\Gamma(n)\Gamma(n+\alpha)}}. \tag{26}$$

Also, here we discuss the raising and lowering operator which is corresponding to the generalized Dirac equation. So, we can factorize the associated Laguerre differential equation for the corresponding problem with respect to the parameter n as follows,

$$A_n^+(z)A_n^-(z)L_n^{(\alpha,1)}(z)=n(n+\alpha-1)L_n^{(\alpha,1)}(z), \tag{27}$$

$$A_n^-(z)A_n^+(z)L_n^{(\alpha,1)}(z)=n(n+\alpha-1)L_n^{(\alpha,1)}(z). \tag{28}$$

So, the differential operators as functions of parameters n will be as,

$$A_n^+(z)=z\frac{d}{dz}-z+n+\alpha-1,$$

$$A_n^-(z)=-z\frac{d}{dz}+n \tag{29}$$

Note that the shape invariance equation (27)and (28) can also be written as the raising and lowering relation,

$$A_n^+(z)L_{n-1}^{(\alpha,1)}(z)=\sqrt{n(n+\alpha-1)}L_n^{(\alpha,1)}(z),$$

$$A_n^-(z)L_n^{(\alpha,1)}(z)=\sqrt{n(n+\alpha-1)}L_{n-1}^{(\alpha,1)}(z). \tag{30}$$

4 Conclusion

In this paper we studied the massless scalar field on the AdS_3 and obtained the Schrödinger like equation. This corresponding hamiltonian factorized in terms of two first order operators which are known raising and lowering operators. These two operators have a index n and m. In order to have shape invariance condition $H_1 = H_2$ or $V_1 = V_2$, two operators must be just factorized in terms of m. The shape invariance conditions lead us to obtained the partner potential and superpotential. Finally, we have shown the raising and lowering operators satisfied to the commutation relation algebra. It may be interesting to do this process for the massive particle in AdS_3 or any arbitrary space.

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