A PROOF OF BEAL’S CONJECTURE

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ABSTRACT. Beal’s Conjecture: The equation $z^\xi = x^\mu + y^\nu$ has no solution in relatively prime positive integers $x, y, z$ with $\mu, \xi$ and $\nu$ odd primes at least 3. A proof of this longstanding conjecture is given.

Beal’s Conjecture: The equation $z^\xi = x^\mu + y^\nu$ has no solution in relatively prime positive integers $x, y, z$ with $\xi, \mu$ and $\nu$ odd primes at least 3. A history of this problem can be found in [1].

Suppose $z^\xi = x^\mu + y^\nu$ is true for any relatively prime positive integers $x, y, z$ and odd primes $\xi, \mu$ and $\nu$ with $\xi, \mu, \nu$ at least 3. When $x, y$ and $z$ are relatively prime, ($z^\xi$)$(x^\xi)$ and ($y^\xi$) are also relatively prime. Then $(z^\xi)^\xi = (x^\xi)^\mu + (y^\xi)^\nu$. That is, suppose $(z^\xi)^\xi = (x^\mu)^\xi + (y^\nu)^\xi$.

The Proof.

We claim the following:

$$x^\mu + y^\nu - z^\xi \equiv 0 \pmod{\xi},$$

and

$$(x^\mu + y^\nu)^\xi - (z^\xi)^\xi \equiv 0 \pmod{\xi^2}$$

To prove the above claims:

Note that by expanding $(x^\mu + y^\nu - z^\xi)^\xi$ using binomial expansion,

$$ (x^\mu + y^\nu - z^\xi)^\xi - ((x^\mu + y^\nu)^\xi - (z^\xi)^\xi) = \sum_{k=1}^{\xi-1} C(\xi, k)(x^\mu + y^\nu)^{\xi-k}(-z^\xi)^k, \quad (1) $$

Again, using binomial expansions for $(x^\mu + y^\nu)^\xi$ and $((x^\mu + y^\nu - z^\xi) + z^\xi)^\xi$.

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\[(x^\mu + y^\nu)^\xi - (z^\xi)^\xi - (x^\mu + y^\nu - z^\xi)^\xi \equiv 0 \pmod{\xi}. \quad (2)\]

The right hand side of equation (2) is divisible by \(\xi\) and hence the left hand side is divisible by \(\xi\). The expansion of \((x^\mu + y^\nu)^\xi - (z^\xi)^\xi\) shows that \((x^\mu + y^\nu)^\xi - (z^\xi)^\xi\) is divisible by \(\xi\) and hence \((x^\mu + y^\nu - z^\xi)^\xi\) is divisible by \(\xi\). Thus
\[x^\mu + y^\nu - z^\xi \equiv 0 \pmod{\xi}. \quad (3)\]

So,
\[(x^\mu + y^\nu - z^\xi)^\xi \equiv 0 \pmod{\xi^\xi}.
\]

Also from equations (2) and (3), and since
\[(x^\mu + y^\nu)^\xi - (z^\xi)^\xi - (x^\mu + y^\nu - z^\xi)^\xi = \xi S,
\]
where \(\xi S\) represents a sum of terms with \((x^\mu + y^\nu - z^\xi)\) as a factor and a multiple of \(\xi\) as coefficient, we have
\[(x^\mu + y^\nu)^\xi - (z^\xi)^\xi \equiv 0 \pmod{\xi^2}. \quad (4)\]

In view of equations (3) and (4), equation (1) gives that
\[z^\xi \equiv 0 \pmod{\xi} \quad (5)\]

and
\[x^\mu + y^\nu \equiv 0 \pmod{\xi}. \quad (6)\]

Hence, in view of equation (3),
\[(z^\xi)^\xi - (x^\mu)^\xi - (y^\nu)^\xi = (x^\mu + y^\nu)^\xi - (x^\mu)^\xi - (y^\nu)^\xi
\]
\[= \sum_{k=1}^{\xi-1} C^r(\xi, k)(x^\mu)^{\xi-k}(y^\nu)^k \equiv 0 \pmod{\xi^\xi}. \quad (7)\]

So,
\[y^\nu \equiv 0 \pmod{\xi} \quad (8)\]

and
\[x^\mu \equiv 0 \pmod{\xi} \quad (9).\]

Thus we get \(x \equiv 0 \pmod{\xi}, y \equiv 0 \pmod{\xi}\) and \(z \equiv 0 \pmod{\xi}\). Hence \(x, y, z\) are not relatively prime and thus proves Beal’s Conjecture.
REFERENCES


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