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A new Pachpatte type dynamic inequality on time scales

Weihong Sheng, Wei Nian Li

Department of Mathematics, BinzhouUniversity, Shandong 256603, P.R.China

Emails: wh-sheng@163.com(W.Sheng), wnli@263.net(W.N.Li)

Abstract. In this paper, using the comparison theorem, we investigate a new Pachpatte type dynamic inequality on time scales, which provides explicit bounds on unknown functions. Our result unifies and extends a continuous inequality and its corresponding discrete analogues.

Keywords: Time scale; Dynamic inequality; Dynamic equation.

1. Introduction

In 1988 Hilger[1] introduced the calculus on time scales inorder to unify the theory of continuous and discrete dynamic systems. In the past few years, motivated by the paper [1], many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the literatures [2-9] and the references cited therein. In this paper, we investigate a new Pachpatte type dynamic inequality on time scales, which unifies and extends a continuous inequality and its corresponding discrete analogues. The obtained inequality can be used as important tools in the study of certain properties of dynamic equations on time scales.

2. Preliminaries

Throughout this paper, a knowledge and understanding of time scales and time scale notation are assumed. For an excellent introduction to the calculus on time scales, we refer the reader to monographs[2,3].

In what follows, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous functions, $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{for all } t \in \mathbb{T}\}$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left—scattered maximum m , then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$;otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_{+} = [0, \infty)$ and $\mathbb{N}_{0} = \{0, 1, 2, \cdots\}$ denotes the set of nonnegative integers. We use the usual conventions that empty sums and products are taken to be 0 and 1respectively.

Lemma 2.1.([2]) *Let* $t_0 \in \mathbb{T}^{\kappa}$ *and* $w : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ *be continuous at* (t, t) *,where* $t \geq t_0$, $t \in \mathbb{T}^{\kappa}$ *with* $t > t_0$. *Assume that* $w^{\Delta}(t, \cdot)$ *is rd--continuous on* $[t_{0,} \sigma(t)]$. *If for any* $\varepsilon > 0$, *there exists a neighborhood U oft, independent of* $\tau \in [t_{0,} \sigma(t)]$ *such that*

$$\left| w(\sigma(t), \tau) - w(s, t) - w^{\Delta}(t, \tau)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right| \quad for\ all\ s \in U ,$$

where w^{Δ} *denotes the derivative of* w *with respect to the first variable, then*

$$g(t) := \int_{t_0}^{t} w(t, \tau) \Delta \tau$$

implies

$$g^{\Delta}(t) = \int_{t_0}^{t} w^{\Delta}(t, \tau) \Delta \tau + w(\sigma(t), t).$$

The following theorem is a foundational result in dynamic inequalities.

Lemma 2.2 (Comparison Theorem[2]). *Suppose* $u, b \in C_{rd}$, $a \in \mathcal{R}^{+}$. *Then*

$$u^{\Delta}(t) \leq a(t) u(t) + b(t), \quad t \geq t_0, t \in \mathbb{T}^{\kappa}$$

implies

$$u(t) \leq u(t_0) e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau)) b(\tau) \Delta \tau, \quad t \geq t_0, \ t \in \mathbb{T}^{\kappa}$$

3. Main results

In this section, we establish a new Pachpatte type dynamic inequality on time scales. For convenience, we always assume that $t \geq t_0$, $t \in \mathbb{T}^{\kappa}$.

Theorem 3.1. *Assume that* $u, a, f, g, p \in C_{rd}$, $u(t)$, $a(t)$, $f(t)$, $g(t)$ *and* $p(t)$ *are nonnegative, and* $a(t)$ *and* $p(t)$ *are non decreasing .If* $w(t, s)$ *is defined asin Lemma 2.1 such that* $w(t, s) \geq 0$ *and* $w^{\Delta}(t, s) \geq 0$ *for* $t, s \in \mathbb{T}$ *with* $s \leq t$ *, then*

$$\begin{aligned} u(t) \leq a(t) + p(t) \Big\{ & \int_{t_0}^{t} f(\tau) u(\tau) \Delta \tau + \int_{t_0}^{t} f(\tau) p(\tau) \left(\int_{t_0}^{\tau} g(s) u(s) \Delta s \right) \Delta \tau \\ & + \int_{t_0}^{t} f(\tau) p(\tau) \left[\int_{t_0}^{\tau} g(s) p(s) \left(\int_{t_0}^{s} w(s, \xi) u(\xi) \Delta \xi \right) \Delta s \right] \Delta \tau \Big\}, \quad t \in \mathbb{T}^{\kappa}, \qquad (I1) \end{aligned}$$

implies $u(t) \le a(t)\left\{1 + p(t)\int_{t_0}^{t} f(\tau)\left[1 + p(\tau)\left(\int_{t_0}^{\tau} e_{fp}(\tau,\sigma(s))\left(f(s) + g(s) + g(s)p(s)G(s)\right)\Delta s\right)\right]\Delta\tau\right\},$

$$t \in \mathbb{T}^{\kappa}, \qquad (I2)$$

where $$A(t)=w(\sigma(t),t)+\int_{t_0}^{t} w^{\Delta}(t,s)\Delta s, \tag{3.1}$$

and $$G(s)=\int_{t_0}^{s} e_{p(f+g+A)}(s,\sigma(\xi))\left[f(\xi)+g(\xi)+A(\xi)\right]\Delta\xi. \tag{3.2}$$

Proof. Let

$$m(t) = \int_{t_0}^{t} f(\tau)u(\tau)\Delta\tau + \int_{t_0}^{t} f(\tau)p(\tau)\left(\int_{t_0}^{\tau} g(s)u(s)\Delta s\right)\Delta\tau$$

$$+\int_{t_0}^{t} f(\tau)p(\tau)\left[\int_{t_0}^{\tau} g(s)p(s)\left(\int_{t_0}^{s} w(s,\xi)u(\xi)\Delta\xi\right)\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.3}$$

Then $m(t_0) = 0$,

$$u(t) \le a(t)+p(t)m(t), \tag{3.4}$$

and

$$m^{\Delta}(t) = f(t)u(t)+f(t)p(t)\int_{t_0}^{t} g(s)u(s)\Delta s + f(t)p(t)\int_{t_0}^{t} g(s)p(s)\left(\int_{t_0}^{s} w(s,\xi)u(\xi)\Delta\xi\right)\Delta s$$

$$\le f(t)\left\{a(t)+p(t)\left[m(t) + \int_{t_0}^{t} g(s)\left[a(s) + p(s)m(s)\right]\Delta s\right.\right.$$

$$\left.\left.+\int_{t_0}^{t}\left(g(s)p(s)\int_{t_0}^{s} w(s,\xi)\left[a(\xi)+p(\xi)m(\xi)\right]\Delta\xi\right)\Delta s\right]\right\}, \quad t \in \mathbb{T}^{\kappa}. \tag{3.5}$$

Setting

$$v(t) = m(t) + \int_{t_0}^{t} g(s)\left[a(s) + p(s)m(s)\right]\Delta s$$

$$+\int_{t_0}^{t}\left(g(s)p(s)\int_{t_0}^{s} w(s,\xi)\left[a(\xi)+p(\xi)m(\xi)\right]\Delta\xi\right)\Delta s, \quad t \in \mathbb{T}^{\kappa}, \tag{3.6}$$

we easily *see that* $v(t_0) = m(t_0) = 0$, $m(t) \le v(t)$.

$$m^{\Delta}(t) \le f(t)\left[a(t)+p(t)v(t)\right] \tag{3.7}$$

and

$$v^{\Delta}(t) = m^{\Delta}(t) + g(t)\left[a(t) + p(t)m(t)\right] + g(t)p(t)\int_{t_0}^{t} w(s,\xi)\left[a(\xi)+p(\xi)m(\xi)\right]\Delta\xi$$

$$= m^{\Delta}(t) + g(t)\left\{a(t) + p(t)\left[m(t)+\int_{t_0}^{t} w(s,\xi)\left[a(\xi)+p(\xi)m(\xi)\right]\Delta\xi\right]\right\}$$

$$\le f(t)a(t)+f(t)p(t)v(t)$$

$$+g(t)\left\{a(t)+\ p(t)\left[v(t)+\int_{t_0}^{t} w(s,\xi)\left[a(\xi)+p(\xi)v(\xi)\right]\Delta\xi\right]\right\},\qquad t\in\mathbb{T}^{\kappa}.\quad (3.8)$$

Define

$$y(t)\ =\ v(t)+\int_{t_0}^{t} w(t,\xi)\left[a(\xi)+p(\xi)v(\xi)\right]\Delta\xi, \qquad (3.9)$$

Then $y(t_0)\ =\ v(t_0)\ =\ 0,\ \ v(t)\le\ y(t),$

$$v^{\Delta}(t)\le\ f(t)p(t)v(t)+\left[\left(\ f(t)+\ g(t)\right)a(t)+\ g(t)p(t)y(t)\right], \qquad (3.10)$$

and

$$\begin{aligned} y^{\Delta}(t)\ &=\ v^{\Delta}(t)+w(\sigma(t),t)\left[a(t)+p(t)v(t)\right]+\int_{t_0}^{t} w^{\Delta}(t,\xi)\left[a(\xi)+p(\xi)v(\xi)\right]\Delta\xi \\ &\le\ f(t)p(t)v(t)+\left[\left(\ f(t)+\ g(t)\right)a(t)+\ g(t)p(t)y(t)\right] \\ &\quad+\left[a(t)+p(t)v(t)\right]\left(w(\sigma(t),t)+\int_{t_0}^{t} w^{\Delta}(t,\xi)\Delta\xi\right) \\ &\le\left[\ f(t)+\ g(t)+A(t)\right]a(t)+\left[\ f(t)+\ g(t)+A(t)\right]p(t)y(t),\qquad t\in\mathbb{T}^{\kappa}. \end{aligned} \qquad (3.11)$$

Using Lemma 2.2, from (3.11) we obtain

$$y(t)\ \ \le\int_{t_0}^{t} e_{p(\ f+g+A)}(t,\sigma(\xi))\left[\ f(\xi)+g(\xi)+A(\xi)\right]a(\xi)\Delta\xi,\qquad t\in\mathbb{T}^{\kappa}. \qquad (3.12)$$

On the other hand, using Lemma 2.2, (3.10) guarantees

$$v(t)\le\int_{t_0}^{t} e_{fp}(t,\sigma(s))\left(\left(\ f(s)+g(s)\right)a(s)+g(s)p(s)y(s)\right)\Delta s,\quad t\in\mathbb{T}^{\kappa}. \qquad (3.13)$$

Substituting (3.12)in (3.13), and noting $a(t)$ is non decreasing, we have

$$v(t)\le\ a(t)\int_{t_0}^{t} e_{fp}(t,\sigma(s))\left(\left(\ f(s)+g(s)\right)+g(s)p(s)G(s)\right)\Delta s,\quad t\in\mathbb{T}^{\kappa}, \qquad (3.14)$$

where $G(s)$ is as defined in (3.2).

Substituting (3.14) in (3.7),setting $t\ =\ \tau$, integrating it from t_0 to t , and noting $m(t_0)\ =\ 0$ and $a(t)$ is non decreasing, we easily obtain

$$m(t)\le\ a(t)\int_{t_0}^{t} f(\tau)\left\{1+\ p(t)\left[\int_{t_0}^{\tau} e_{fp}(\tau,\sigma(s))\left(\ f(s)+g(s)+g(s)p(s)G(s)\right)\Delta s\right]\right\}\Delta\tau,$$
$$t\in\mathbb{T}^{\kappa}.\quad (3.15)$$

It is easy to see that the desired inequality (I2) follows from(3.4) and (3.15). This completes the proof of Theorem 3.1.

Remark 3.2. In Theorem 3.1, assume that $a(t) = u_0$, $p(t) = 1$, $u_0 \geq 0$ is a constant. We easily obtain Theorem3.5 in [6]. Therefore, Theorem 3.1 is the generalization of Theorem 3.5 in [6].

Remark 3.3. Let $w(t,s) = w(s)$ in theorem 3.1. If $\mathbb{T} = \mathbb{R}$ then the inequality established in Theorem3.1 reduces to the inequality established by Pachpatte in [10,Theorem 1.7.3(ii)]. If $\mathbb{T} = \mathbb{Z}$, then from Theorem3.1, we easily obtain Theorem 1.4.7(v) in [11].

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