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# Permanence and extinction for a delayed periodic predator-prey system

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## Abstract

In this paper, the permanence, extinction and periodic solution of a delayed periodic predator-prey system with Holling type IV functional response and stage structure for prey is studied. By means of comparison theorem, some sufficient and necessary conditions are derived for the permanence of the system.

**Key words:** Predator-prey system; Holling type IV functional response; Permanence; Extinction; Stage structure.

## 1. Introduction

Since the predator-prey system has established and were accepted by many scientists and now it became the most important means to explain the ecological phenomenon. For many years, a lot of extensive research results were made in mathematical biology and Mathematical ecology, during this time predator-prey system has played an important role in theses research field of mathematical biology and mathematical ecology. Still now many research work mostly discussed permanence(or uniformly persistence) and global stability of periodic Lotka-Volterra predator-prey systems[1-14].

As we well known, functional response and stage structure population system is one of the most important class of model which is discussed widely in mathematical biology and mathematical ecology. Recently, predator-prey population dynamical systems with stage structure and functional response have been discussed by many authors, see for example[1-12] and references cited therein.

In [2], the authors have studied the following ratio-dependent predator-prey system

$$\dot{x}_1(t) = ax_2(t) - r_1x_1(t) - bx_1(t),$$
$$\dot{x}_2(t) = bx_1(t) - b_1x_2^2(t) - \frac{a_1x_2(t)y(t)}{my(t) + x_2(t)},$$
$$\dot{x}_3(t) = y(t)\left(-r + \frac{a_2x_2(t)}{my(t) + x_2(t)}\right),$$

where $x_1(t)$ represents the density of immature individuals preys at time t, and$x_2(t)$represents the density of mature individuals preys at time $t$, $y(t)$ represents the density of predator at time $t$. The authors obtained sufficient conditions for the uniform persistence, impermanence and the global asymptotic stability of nonnegative equilibria of the model.

In[4], the authors considered the following delayed predator-prey system with Beddington-De Angelis functional and stage structure for prey:

$$\begin{aligned}\dot{x}_1(t) &= b(t)x_2(t) - d_1(t)x_1(t)\\ &\quad -b(t-\tau_1)\exp\left(-\int_{t-\tau_1}^{t} d_1(s)ds\right)x_2(t-\tau_1) - \frac{c_1(t)x_1(t)y(t)}{e(t) + \beta(t)x_1(t) + \gamma(t)y(t)},\\ \dot{x}_2(t) &= b(t-\tau_1)\exp\left(-\int_{t-\tau_1}^{t} d_1(s)\,ds\right)x_2(t-\tau_1) - a(t)x_2^2(t),\\ \dot{y}(t) &= y(t)\left(-d_2(t) + \frac{c_2(t)x_1(t-\tau_2)}{e(t) + \beta(t)x_1(t-\tau_2) + \gamma(t)y(t-\tau_2)} - q(t)y(t)\right),\end{aligned}$$

where $x_1(t)$ and $x_2(t)$denote the densities of immature and mature preys at time $t$, respectively; $y(t)$ represent the density of predators at time $t$. The authors obtained sufficient conditions for the permanence, extinction and periodic solution of the system.

In[7], the authors have studied the following delayed predator-prey system with Holling type IV functional and stage structure for predators:

$$\dot{x}(t) = x(t)\left(r(t) - a(t)x(t-\tau_1) - \frac{b(t)y_2(t)}{kx^2(t) + x(t) + a^*}\right),$$
$$\dot{y}_1(t) = \frac{k(t)b(t)x(t-\tau_2)y_2(t-\tau_2)}{kx^{2(t-\tau_2)} + x(t-\tau_2) + a^*} - \big(D(t) + v_1(t)\big)y_1(t) - k_1(t)y_1^2(t),$$
$$\dot{y}_2(t) = D(t)y_1(t) - v_2(t)y_2(t),$$

where$x(t)$ is the density of the prey population at time $t$, $y_1(t)$ is the density of immature predators at time $t$, $y_2(t)$ is the density of mature predators at time $t$. The authors obtained sufficient conditions for the existence of multiple positive periodic solutions of the system.

Motivated by the above works, in this paper, we study the following delayed predator- prey system with Holling type IV response function and stage structure for prey

$$
\begin{aligned}
\dot{x}_1(t) &= r(t)x_2(t) - B(t)x_1(t) - d_1(t)x_1^2(t),\\
\dot{x}_2(t) &= B(t)x_1(t) - d_2(t)x_2^2(t) - \frac{a_1(t)x_2(t)y(t-\tau_1)}{kx_2^2(t)+x_2(t)+a^*},\\
\dot{y}(t) &= y(t)\left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2)+x_2(t-\tau_2)+a^*} - d(t)y(t)\right),
\end{aligned}
\tag{1.1}
$$

where $x_1(t)$ represents the density of immature individuals preys at time $t$, and $x_2(t)$represents the density of mature individuals preys at time $t$, $y(t)$ represents the density of predator at time $t$. $r(t) > 0$ represents the birth rate of the immature prey at time $t$, $B(t) = b(t-\tau_1)\exp\left(-\int_{t-\tau_1}^{t} d_1(s)ds\right)$ represents the number of immature preys that were born at time $t-\tau_1$ which is still survive at time t and are transferred from the immature stage to the mature stage at time $t, d_1(t), d_2(t)$ and $d_3(t)$ are the death rates of the immature prey, mature prey, and predator at time $t$ respectively, $a_1(t)$ is the capturing rate of the predator, $a_2(t)/a_1(t)$ is the rate of conversion of nutrients into reproduction of the predator, constant, the item $-d(t)y(t)$ represents the dynamics of predator $y$ to incorporate the negative feedback of predator crowding. $r(t), d_i(t)(i=1,2,3)$, $B(t), d(t), a_i(t)$ $(i=1,2)$ are all continuously positive $\omega$ −periodic functions, $\tau_i(i=1,2), k, a^*$ are positive constants.

Due to the biological meaning of the model, the initial conditions for system (1.1) take the form

$$
\begin{cases}
\left(x_{1(\theta)}, x_2(\theta), y(\theta)\right) = \left(\varphi_1(\theta), \varphi_2(\theta), \psi(\theta)\right) \in C_+ =: C([-\tau_{\max}, 0], R_+^3),\\
\qquad \varphi_1(0) > 0, \quad \varphi_2(0) > 0, \quad \psi(0) > 0
\end{cases}
\tag{1.2}
$$

where $\tau_{max} = \max\{\tau_1, \tau_2\}$ , $R_+^3 = \{(x,y,z) | x \geq 0, y \geq 0, z \geq 0\}$.

If the term $d(t) = 0$ then system (1.1) become the following system

$$
\begin{aligned}
\dot{x}_1(t) &= r(t)x_2(t) - B(t)x_1(t) - d_1(t)x_1^2(t),\\
\dot{x}_2(t) &= B(t)x_1(t) - d_2(t)x_2^2(t) - \frac{a_1(t)x_2(t)y(t-\tau_1)}{kx_2^2(t)+x_2(t)+a^*},\\
\dot{y}(t) &= y(t)\left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2)+x_2(t-\tau_2)+a^*}\right).
\end{aligned}
\tag{1.3}
$$

The system (1.3) also satisfies the initial condition (1.2).

Throughout this paper, for a continuous $\omega$ −periodic function $f(t)$, we set

$$
A\left(f(t)\right) = \frac{1}{\omega}\int_0^{\omega} f(t)dt, \quad f^M = \max_{t\in[0,\omega]} f(t), \quad f^L = \min_{t\in[0,\omega]} f(t).
$$

The organization of this paper is as follows. In the next section we will prove positivity and boundedness of solutions for system (1.1). In section 3, we present the main results in this paper. In section 4, give the proof of the main results.

## 2. Positivity and boundedness of solutions

In this section, we will prove that the solutions of system (1.1) under initial value condition (1.2) are positive and ultimately bounded.

**Theorem 2.1.** The solutions of system (1.1) with initial condition (1.2) are positive for all $t \geq 0$.

**Proof.** Suppose that $\left(x_1(t), x_2(t), y(t)\right)$ is any solution of (1.1) with initial condition (1.2). From system (1.1) we get

$$\begin{aligned} &\dot{x}_1(t)|_{x_1=0} = r(t)x_2(t) > 0, \quad for \quad x_2(t) > 0, \\ &\dot{x}_2(t)|_{x_2=0} = B(t)x_1(t) > 0, \quad for \quad x_1(t) > 0, \\ &\dot{y}(t)|_{y=0} = y(0)\exp\int_0^t \left[-d_3(s) + \frac{a_2(s)x_2(s-\tau_2)}{kx_2^2(s-\tau_2) + x_2(s-\tau_2) + a^*} - d(s)y(s)\right] ds > 0. \end{aligned}$$

**Theorem 2.2.** The solutions of systems (1.1) with initial condition (1.2) are ultimately bounded.

**Proof.** Suppose that $\left(x_1(t), x_2(t), y(t)\right)$ is any solution of (1.1) with initial condition (1.2). Defining the function

$$w(t) = x_1(t) + x_2(t) + y(t)$$

and calculating the derivative of $w(t)$ along the positive solutions of system (1.1), we have

$$\begin{aligned} \dot{w}(t) = \dot{x}_1(t) + \dot{x}_2(t) + \dot{y}(t) \leq r^M x_2(t) - B^L x_1(t) - d_1^L(t)x_1^2(t) + B^M x_1(t) \\ -d_2^L x_2^2(t) - d_3^L y(t) + a_2^M y(t) - d^L y^2(t) \end{aligned}$$

Then

$$\dot{w}(t) + \rho_1 w(t) \leq B^M x_1(t) - d_1^L(t)x_1^2(t) + 2r^M x_2(t) - d_2^L x_2^2(t) + a_2^M y(t) - d^L y^2(t),$$

where $\rho_1 = \min\{r^M, B^L, d_3^L\}$. Then there exists a positive number $\rho_2$ such that

$$\dot{w}(t) + \rho_1 w(t) \leq \rho_2$$

which yields

$$w(t) \leq \frac{\rho_2}{\rho_1} + \left(w(0) - \frac{\rho_2}{\rho_1}\right) e^{-\rho_1 t}.$$

This implies that any positive solutions of system (1.1) is ultimately bounded. This completes the proof.

## 3. Main results

In this section we introduce some lemmas and state the main results.

**Definition 3.1** System (1.1) is said to be permanent if there exist positive constants $m$, $M$ and $T_0$, such that each positive solution $(x_1(t), x_2(t), y(t))$ of system (1.1) with any positive initial value $\varphi$, fulfill $m \leq x_i(t) \leq M$ $(i = 1,2), m \leq y(t) \leq M$ for all $t \geq T_0$, where $T_0$ may depend on $\varphi$.

**Lemma 3.1([15])** If $a(t), b(t), d(t)$ and $f(t)$ are all $\omega$ −periodic, then system

$$\begin{aligned} &\dot{x}_1(t) = a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t), \\ &\dot{x}_2(t) = c(t)x_1(t) - f(t)x_2^2(t) \end{aligned} \tag{3.1}$$

has a positive $\omega$ −periodic solution $\left(x_1^*(t), x_2^*(t)\right)$ which is globally asymptotically stable with respect to $R_+^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$.

**Lemma 3.2([16])** If $b(t)$ and $a(t)$ are all $\omega$ −periodic, and if $A_\omega(b) > 0$ and $A_\omega(a) > 0$ for all $t \in R$, then system

$$\dot{x}(t) = x(t)\left(b(t) - a(t)x(t)\right), \tag{3.2}$$

has a positive $\omega$ −periodic solution which is globally asymptotically stable.

**Theorem 3.1.** System (1.1) is permanent if and only if

$$A\left(-d_3(t) + \frac{a_2(t)x_2^*(t-\tau_2)}{kx_2^{*2}(t-\tau_2) + x_2^*(t-\tau_2) + a^*}\right) > 0 \tag{3.3}$$

where $(x_1^*(t), x_2^*(t))$ is the unique positive periodic solution of system (3.1) given by Lemma 3.1.

As a direct corollary of [17, Theorem 2], from Theorem 2.1, we have

**Corollary 3.1.** If the assumption (3.3) hold, then system (1.1) has at least one positive $\omega$ −periodic solution.

From the proof of Theorem 3.1, we also have

**Theorem 3.2.** If the assumption (3.3) hold, then system (7) is permanent and system (1.3) has at least one positive $\omega$ −periodic solution.

**Corollary 3.2.** Assume that

$$A\left(-d_3(t) + \frac{a_2(t)x_2^*(t-\tau_2)}{kx_2^{*2}(t-\tau_2) + x_2^*(t-\tau_2) + a^*}\right) \le 0 \tag{3.4}$$

where $(x_1^*(t), x_2^*(t))$ is the unique positive periodic solution of system (3.1) given by Lemma 3.1, then any solution of system (1.1) and (1.3) with initial condition (1.2) satisfies

$$\lim_{t\to+\infty} y(t) = 0. \tag{3.5}$$

## 4. Proof of the main results

In this section we will state and prove four propositions to prove Theorem 3.1.

**Proposition 4.1.** There exist positive constants $M_x$ and $M_y$, such that

$$\limsup_{t\to\infty} x_i(t) \le M_x, \quad \limsup_{t\to\infty} y(t) \le M_y$$

for all solution of system (1.1) with initial condition (1.2).

**Proof.** Obviously, $R_+^3 = \{(x_1(t), x_2(t), y(t)) | x_i(t) \ge 0, y(t) \ge 0\}$ is a positively invariant set of system (1.1) with initial condition (1.2), we have

$$\begin{aligned}\dot{x}_1(t) &= r(t)x_2(t) - B(t)x_1(t) - d_1(t)x_1^2(t),\\ \dot{x}_2(t) &\le B(t)x_1(t) - d_2(t)x_2^2(t),\end{aligned}$$

By Lemma 3.1, the following auxiliary equation:

$$\begin{aligned}\dot{u}_1(t) &= r(t)u_2(t) - B(t)u_1(t) - d_1(t)u_1^2(t),\\ \dot{u}_2(t) &= B(t)u_1(t) - d_2(t)u_2^2(t),\end{aligned} \tag{4.1}$$

has unique globally asymptotically stable positive $\omega$ −periodic solution $(x_1^*(t), x_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of with $(u_1(0), u_2(0)) = (x_1(t), x_2(t))$. In view of the vector comparison method, we have

$$x_i(t) \le u_i(t) (i = 1,2), \quad t \ge 0. \tag{4.2}$$

Moreover, from the global attractivity of$\left(x_1^*(t), x_2^*(t)\right)$, for every given $\varepsilon(0<\varepsilon<1)$, there exists a $T_1>0$, such that

$$|u_i(t)-x_i^*(t)|<\varepsilon, \quad t\geq T_1 \tag{4.3}$$

from (4.2) and (4.3) we have

$$\lim_{t\to+\infty}\sup x_i(t)\leq M_x,$$

where $M_x=\max_{0\leq t\leq\omega}\{\backslash x_i^*(t)+\varepsilon,\ i=1,2\}$.

In addition, from the third equation of system (1.1) we get

$$\dot{y}(t)\leq y(t)\big(a_2(t)-d(t)y(t)\big).$$

Consider the following auxiliary equation:

$$\dot{v}(t)\leq v(t)\big(a_2(t)-d(t)y(t)\big), \tag{4.4}$$

it follows from Lemma 3.2 that equation (4.4) has a unique positive $\omega$ −periodic solution $y^*(t)>0$ which is globally asymptotically stable. Similarly to the above analysis, there exists a $T_2>T_1$ such that for above $\varepsilon$, one has

$$y(t)<y^*(t)+\varepsilon, \quad t\geq T_2.$$

This completes the proof of proposition 4.1.

**Proposition 4.2.** There exist positive constant $m_{ix}<M_x$, such that

$$\lim_{t\to+\infty}\inf x_i(t)\geq m_{ix}, \quad i=1,2.$$

**Proof.** By proposition 4.1, there exists $T_2>0$ such that

$$0<y(t)\leq M_y, \quad t\geq T_2.$$

Hence, from system (1.1), we have

$$\begin{aligned}\dot{x}_1(t)&=r(t)x_2(t)-B(t)x_1(t)-d_1(t)x_1^2(t),\\ \dot{x}_2(t)&\leq B(t)x_1(t)-\left(d_2(t)+\frac{a_1(t)}{a^*}M_y\right)x_2^2(t),\end{aligned}$$

for $t\geq T_2$. By lemma 3.1, the following auxiliary equation

$$\begin{aligned}\dot{u}_1(t)&=r(t)u_2(t)-B(t)u_1(t)-d_1(t)u_1^2(t),\\ \dot{u}_2(t)&\leq B(t)u_1(t)-\left(d_2(t)+\tfrac{a_1(t)}{a^*}M_y\right)u_2^2(t),\end{aligned} \tag{4.5}$$

has a unique globally attractive positive $\omega$ −periodic solution $\left(x_1^*(t), x_2^*(t)\right)$. Let $\left(u_1(t), u_2(t)\right)$ be the solution of (4.5) with $\left(u_1(T_2), u_2(T_2)\right)=(\left(x_1(T_2), x_2(T_2)\right)$, by comparison method, we have

$$x_i(t)\geq u_i(t)(i=1,2), \quad t\geq T_2. \tag{4.6}$$

Again from the globally attractivity of $\left(x_1^*(t), x_2^*(t)\right)$, there exists a $T_3>T_2$, such that

$$|u_i(t)-x_i^*(t)|<\tfrac{x_i^*(t)}{2}, (i=1,2) \quad t\geq T_3. \tag{4.7}$$

Equation (4.7) combine with (4.6) leads to

$$x_i(t) > m_{ix} = \min_{0\le t\le\omega}\left\{\frac{x_i^*(t)}{2},\ i=1,2\right\} \quad t > T_3.$$

Therefore,

$$\lim_{t\to+\infty} \inf x_i(t) \ge m_{ix}, \quad i=1,2.$$

This completes the proof of proposition 4.2.

**Proposition 4.3.** Suppose that (3.3) holds, then there exists a positive constant $m_y$ such that any solution $\left(x_1(t), x_2(t), y(t)\right)$ of system (1.1) with initial condition (1.2) satisfies

$$\lim_{t\to+\infty} \sup y(t) \ge m_y. \tag{4.8}$$

**Proof.** By (3.3), we can choose constant $\varepsilon_0 > 0$ (without loss of generality, we may assume that $\varepsilon_0 < \frac{1}{2}\min_{t\in[0,\omega]}\{x_1^*(t)\}$, where $\left(x_1^*(t), x_2^*(t)\right)$ is the unique positive periodic solution of system (3.1) such that

$$A\left(\psi_{\varepsilon_0}(t)\right) > 0, \tag{4.9}$$

where

$$\psi_{\varepsilon_0}(t) = -d_3(t) + \frac{a_2(t)(x_2^*(t-\tau_2)-\varepsilon_0)}{k\left(x_2^{*^2}(t-\tau_2)-\varepsilon_0\right)+(x_2^*(t-\tau_2)-\varepsilon_0)+a^*} - d(t)\varepsilon_0.$$

Consider the following equation with a parameter $\alpha$,

$$\begin{aligned}\dot{x}_1(t) &= r(t)x_2(t) - B(t)x_1(t) - d_1(t)x_1^2(t),\\ \dot{x}_2(t) &= B(t)x_1(t) - \left(d_2(t) + \frac{2a_1(t)}{a^*}\alpha\right)x_2^2(t).\end{aligned} \tag{4.10}$$

By Lemma 3.1, (4.10) has a unique positive $\omega$ −periodic solution $\left(x_{1\alpha}^*(t), x_{2\alpha}^*(t)\right)$ which is globally attractive. Let $\left(x_{1\alpha}(t), x_{2\alpha}(t)\right)$ be solution of (4.10) with initial condition $x_{i\alpha}(0) = x_i^*(0)(i=1,2)$, where $\left(x_1^*(t), x_2^*(t)\right)$ is the unique positive periodic solution of system (3.1). Hence, for above $\varepsilon_0$, there exists a sufficiently large $T_4 > T_3$ such that

$$|x_{i\alpha}(t) - x_{i\alpha}^*(t)| < \frac{\varepsilon_0}{4}, \quad (i=1,2), \quad for \quad t \ge T_4.$$

By the continuity of the solution in the parameter, we have $x_{i\alpha}(t) \to x_{i\alpha}^*(t)$ uniformly in $[T_4, T_4+\omega]$ as $\alpha \to 0$. Hence, for $\varepsilon_0 > 0$, there exists a $\alpha_0 = \alpha_0(\varepsilon_0) > 0$ such that

$$|x_{i\alpha}(t) - x_{i\alpha}^*(t)| < \frac{\varepsilon_0}{4}, \quad (i=1,2), \quad for \quad t \in [T_4, T_4+\omega].$$

So, we have

$$|x_{i\alpha}^*(t) - x_i^*(t)| < \frac{\varepsilon_0}{2}, \quad (i=1,2), \quad for \quad t \in [T_4, T_4+\omega].$$

Note that $x_{i\alpha}^*(t)$ and $x_i^*(t)$ are all $\omega$ −periodic, hence

$$|x_{i\alpha}^*(t)-x_i^*(t)|<\frac{\varepsilon_0}{2},\quad (i=1,2),\quad for\quad t\geq 0, 0<\alpha<\alpha_0.$$

Choosing a constant $\alpha_1(0<\alpha_1<\alpha_0, 2\alpha_1<\varepsilon_0)$, we have

$$x_{i\alpha_1}^*(t)\geq x_i^*(t)-\frac{\varepsilon_0}{2},\quad (i=1,2),\quad for\quad t\geq 0. \tag{4.11}$$

Suppose that (4.8) is not true, then there exists $\phi\in R_+^3$ such that

$$\lim_{t\to+\infty}\sup y(t,\phi)<\alpha_1,$$

where $\big(x_1(t,\phi),x_2(t,\phi),y(t,\phi)\big)$ is the solution of system (1.1) with initial condition $\big(x_1(\theta),\ x_2(\theta),$ $y\theta=\phi\theta,\quad \theta\in-\tau,0.$ So there exists $T5>T4$ such that

$$y(t,\phi)<2\alpha_1<\varepsilon_0,\quad t\geq T_5. \tag{4.12}$$

By applying (4.12), from system (1.1) it follows that for all $t\geq T_6\geq T_5+\tau_1$,

$$\begin{aligned}\dot{x}_1(t,\phi)&=r(t)x_2(t,\phi)-B(t)x_1(t,\phi)-d_1(t)x_1^2(t,\phi),\\ \dot{x}_2(t,\phi)&\geq B(t)x_1(t,\phi)-\left(d_2(t)+\frac{2a_1(t)}{a^*}\alpha_1\right)x_2^2(t,\phi),\end{aligned} \tag{4.13}$$

Let $\big(u_{11}(t),u_2(t)\big)$ be the solution of (4.10) with $\alpha=\alpha_1$ and

$$\big(u_1(T_6),u_2(T_6)\big)=\big(x_1(T_6,\phi),x_2(T_6,\phi)\big),$$

then

$$x_i(t,\phi)\geq u_i(t),\qquad (i=1,2),\quad t\geq T_6.$$

By the global asymptotic stability of $(x_{1\alpha_1}^*(t),x_{2\alpha_1}^*(t))$, for the given $\varepsilon=\frac{\varepsilon_0}{2}$, there exists $T_7\geq T_6$, such that

$$\left|u_i(t)-x_{i\alpha_1}^*(t)\right|<\frac{\varepsilon_0}{2},\quad (i=1,2),\quad t\geq T_7.$$

So,

$$x_i(t,\phi)\geq u_i(t)>x_{i\alpha_1}^*(t)-\frac{\varepsilon}{2},\quad (i=1,2),\quad t\geq T_7,$$

and hence, by using (4.11), it follows

$$x_i(t,\phi)\geq x_i^*(t)-\varepsilon_0,\quad (i=1,2),\quad t\geq T_7. \tag{4.14}$$

Therefore, by using (4.12) and (4.14), for $t\geq T_7+\tau_2$ it follows:

$$\begin{aligned}\dot{y}(t,\phi)&\geq\left(-d_3(t)+\frac{a_2(t)(x_2^*(t-\tau_2)-\varepsilon_0)}{k\big(x_2^{*2}(t-\tau_2)-\varepsilon_0\big)+(x_2^*(t-\tau_2)-\varepsilon_0)+a^*}-d(t)\varepsilon_0\right)\\ &=\psi_{\varepsilon_0}(t)y(t,\phi).\end{aligned}$$

Integrating above inequality from $T_7+\tau_2$ to $t$ yields

$$y(t,\phi)\geq y(T_7+\tau_2,\phi)\exp\int_{T_7+\tau_2}^{t}\psi_{\varepsilon_0}(t)\,dt.$$

Thus, from (4.9) it follows that $y(t,\phi) \to +\infty$ as $t \to +\infty$. It is a contradiction. This completes the proof of Proposition 3.

**Proposition 4.4.** Suppose that (3.3) holds, then there exists a positive constant $\tau_y$ such that any solution $(x_1(t), x_2(t), y(t))$ of system (1.1) with initial condition (1.2) satisfies

$$\lim_{t\to+\infty}\inf y(t) \geq \tau_y. \tag{4.15}$$

**Proof.** Suppose that (4.15) is not true, then there exists a sequence$\{\phi_m\} \in R_+^3$, such that

$$\lim_{t\to+\infty}\inf y(t,\phi_m) < \frac{m_y}{(m+1)^2}, \quad m = 1,2,....$$

On the other hand, by Proposition 4.3, we have

$$\lim_{t\to+\infty}\sup y(t,\phi_m) > m_y, \quad m = 1,2,....$$

Hence, there are time sequences $\left\{s_q^{(m)}\right\}$ and $\left\{t_q^{(m)}\right\}$ satisfying

$$0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots,$$

$$s_q^{(m)} \to +\infty, \quad t_q^{(m)} \to +\infty \text{ as } q \to +\infty.$$

And

$$y\left(s_q^{(m)},\phi_m\right) = \frac{m_y}{m+1}, \quad y\left(t_q^{(m)},\phi_m\right) = \frac{m_y}{(m+1)^2},$$

$$\frac{m_y}{(m+1)^2} < y(t,\phi_m) < \frac{m_y}{m+1}, \quad t \in \left(s_q^{(m)}, t_q^{(m)}\right).$$

By proposition 4.1, for a given positive integer $m$, there is a $T_1^{(m)} > 0$, such that for all $t > T_1^{(m)} y(t,\phi_m) < M_y$.

Because of $s_q^{(m)} \to +\infty$ as $q \to +\infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \geq K^{(m)}$, hence

$$\dot{y}(t,\phi_m) \geq y(t,\phi_m)\left(-d_3(t) - d(t)M_y\right) \tag{4.16}$$

for $t \in \left[s_q^{(m)}, t_q^{(m)}\right]$, $q \geq K^{(m)}$. Integrating (4.16) from $s_q^{(m)}$ to $t_q^{(m)}$ yields

$$y\left(t_q^{(m)},\phi_m\right) \geq y\left(s_q^{(m)},\phi_m\right)\exp\left\{\int_{s_q^{(m)}}^{t_q^{(m)}}\left(-d_3(t) - d(t)M_y\right)dt\right\}$$

or

$$\int_{s_q^{(m)}}^{t_q^{(m)}} \left(d_3(t) + d(t)M_y\right) dt \geq \ln(m+1) \quad \text{for} \quad q \geq K^{(m)}$$

thus from the boundedness of $d_3(t) + d(t)M_y$, we have

$$t_q^{(m)} - s_q^{(m)} \to +\infty \quad \text{as} \quad m \to +\infty, \ q \geq K^{(m)} \tag{4.17}$$

By (4.9) and (4.17), there are constants $P > 0$ and $N_0 > 0$, such that

$$\frac{m_y}{m+1} < \alpha_1 < \varepsilon_0, \ t_q^{(m)} - s_q^{(m)} > 2P, \tag{4.18}$$

and

$$\int_0^{\alpha} \psi_{\varepsilon_0}(t)\, dt > 0$$

for $m \geq N_0, q \geq K^{(m)}$ and $\alpha \geq P$. (4.18) implies that

$$y(t, \phi_m) < \alpha_1 < \varepsilon_0, \quad t \in \left[s_q^{(m)}, t_q^{(m)}\right] \tag{4.19}$$

for $m \geq N_0$, $q \geq K^{(m)}$. In addition, for $t \in \left[s_q^{(m)}, t_q^{(m)}\right]$, we have

$$\dot{x}_1(t, \phi_m) = r(t)x_2(t, \phi_m) - B(t)x_1(t, \phi) - d_1(t)x_1^2(t, \phi_m),$$
$$\dot{x}_2(t, \phi_m) \geq B(t)x_1(t, \phi_m) - \left(d_2(t) + \frac{2a_1(t)}{a^*}\alpha_1\right)x_2^2(t, \phi_m),$$

Let $u_1(t), u_2(t)$ be the solution of (4.10) with $\alpha = \alpha_1$ and $u_i\left(s_q^{(m)}\right) = x_i\left(s_q^{(m)}, \phi_m\right)(i = 1,2)$, then bye applying comparison theorem, we have

$$x_i\left(t_q, \phi_m\right) \geq u_i(t), \qquad (i = 1,2), \qquad t \in \left[s_q^{(m)}, t_q^{(m)}\right],$$

Further, by using Proposition 4.1 and 4.2, there exists an enough large $K_1^{(m)} > K_2^{(m)}$ such that

$$\tau_{ix} < x_i\left(s_q^{(m)}, \phi_m\right) < M_x$$

for $q \geq K_1^{(m)}$. For $\alpha = \alpha_1$, (4.10) has a unique positive $\omega$ −periodic solution $\left(x_{1\alpha_1}^*(t), x_{2\alpha_1}^*(t)\right)$ which is globally asymptotically stable. In addition, by the periodicity of (4.10), the periodic solution $\left(x_{1\alpha_1}^*(t), x_{2\alpha_1}^*(t)\right)$ is uniformly asymptotically stable with respect to compact set $\Omega = \{x | \tau_{ix} < x < M_x\}$. Hence, for the given $\varepsilon_0$ in Proposition 4.3, there exists $T_0 = T_0(\varepsilon_0) > P + \tau_2$, which is independent of $m$ and $q$, such that

$$u_i(t) > x_{i\alpha_1}^*(t) - \tfrac{\varepsilon_0}{2}, (i = 1,2), \qquad \text{as} \qquad t > T_0 + s_q^{(m)}.$$

Thus by using (4.11),

$$u_i(t) > x_i^*(t) - \varepsilon_0, (i = 1,2), \quad \text{as} \quad t > T_0 + s_q^{(m)}.$$

By (4.17), there exists a positive integer $N_1 \geq N_0$ such that $t_q^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2(P + \tau_2)$ for $m \geq N_1$ and $q \geq K_1^{(m)}$. So, we have

$$x_i(t, \phi_m) > x_i^*(t) - \varepsilon_0, (i = 1,2), \quad as \quad t \in \left[T_0 + s_q^{(m)}, t_q^{(m)}\right], \tag{4.20}$$

as $m \geq N_1$ and $q \geq K_1^{(m)}$. Hence, by using (4.19) and (4.20), from the third equation of system (1.1), one has

$$\dot{y}(t, \phi_m) \geq \psi_{\varepsilon_0}(t) y(t, \phi_m), \quad t \in \left[T_0 + s_q^{(m)} + \tau_2, t_q^{(m)}\right],$$

Integrating the above inequality from $T_0 + s_q^{(m)} + \tau_2$ to $t_q^{(m)}$ leads to

$$y\left(t_q^{(m)}, \phi_m\right) \geq y\left(T_0 + s_q^{(m)} + \tau_2, \phi_m\right) \exp\left\{\int_{T_0 + s_q^{(m)} + \tau_2}^{t_q^{(m)}} \psi_{\varepsilon_0}(t)\, dt\right\},$$

that is,

$$\frac{m_y}{(1+m)^2} \geq \frac{m_y}{(1+m)^2} \exp\left\{\int_{T_0 + s_q^{(m)} + \tau_2}^{t_q^{(m)}} \psi_{\varepsilon_0}(t)\, dt\right\} > \frac{m_y}{(1+m)^2},$$

This is contradiction. This completes the proof of Proposition 4.4.

**Proof of theorem 3.1.** The sufficiency of this theorem 3.1 now follows from Proposition 4.1-4.4. To prove the necessity of theorem 3.1, we will show that

$$\lim_{t \to +\infty} y(t) = 0.$$

Under the following condition

$$A\left(-d_3(t) + \frac{a_2(t) x_2^*(t - \tau_2)}{k x_2^{*2}(t - \tau_2) + x_2^*(t - \tau_2) + a^*}\right) \leq 0 \tag{4.21}$$

In fact, by (4.21), for every given positive constant $\varepsilon (\varepsilon < 1)$, there exists $\varepsilon_1 > 0 (0 < \varepsilon_1 < \varepsilon)$ and $\varepsilon_0 > 0$ such that

$$A\left(-d_3(t) + \frac{a_2(t)(x_2^*(t - \tau_2) + \varepsilon_1)}{k\left(x_2^{*2}(t - \tau_2) + \varepsilon_1\right) + (x_2^*(t - \tau_2) + \varepsilon_1) + a^*} - d(t)\, \varepsilon\right) \leq -\frac{\varepsilon}{2} A\big(d(t)\big) < -\varepsilon_0.$$

Since

$$\dot{x}_1(t) = r(t) x_2(t) - B(t) x_1(t) - d_1(t) x_1^2(t),$$
$$\dot{x}_2(t) \leq B(t) x_1(t) - d_2(t) x_2^2(t).$$

Hence, for above $\varepsilon_1$ there exists a $T^{(1)} > 0$ such that

$$x_i(t) < x_i^*(t) + \varepsilon_1, (i = 1,2), \qquad \text{for} \quad t \geq T^{(1)}. \tag{4.22}$$

It follows from (4.22) that for $t \geq T^{(1)} + \tau_2$,

$$A\left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2) + x_2(t-\tau_2) + a^*} - d(t)\varepsilon\right) < -\varepsilon_0.$$

First, we show that there exists $T^{(2)} > T^{(1)} + \tau_2$, such that $y\left(T^{(2)}\right) < \varepsilon$. Otherwise, we have

$$\begin{aligned}
&\varepsilon \leq y(t) \\
&\leq y\left(T^{(1)} + \tau_2\right) \exp\left\{\int_{T^{(1)}+\tau_2}^{t} \left(-d_3(s) + \frac{a_2(s)x_2(s-\tau_2)}{kx_2^2(s-\tau_2) + x_2(s-\tau_2) + a^*} - d(s)\varepsilon\right) ds\right\} \\
&\leq y\left(T^{(1)} + \tau_2\right) \exp\{-\varepsilon_0(t - (T^{(1)} + \tau_2))\} \to 0
\end{aligned}$$

as $t \to +\infty$. So, $\varepsilon \geq 0$, which is a contradiction.

Second, we show that

$$y(t) \leq \varepsilon \exp\{M(\varepsilon)\omega\} \text{ for } \quad t \geq T^{(2)}, \tag{4.23}$$

where

$$M(\varepsilon) = \max_{t\in[0,\omega]} \left\{d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2) + x_2(t-\tau_2) + a^*} + d(t)\varepsilon\right\}$$

is bounded for $\varepsilon \in [0,1]$. Otherwise, there exists a $T^{(3)} > T^{(2)}$ such that

$$y\left(T^{(3)}\right) > \varepsilon \exp\{M(\varepsilon)\omega\}.$$

By the continuity of $y(t)$, there must exists $T^{(4)} \in \left(T^{(2)}, T^{(3)}\right)$ such that $y\left(T^{(4)}\right) = \varepsilon$ and $y(t) > \varepsilon$ for $t \in (T^{(4)}, T^{(3)}]$.

Let $P_1$ be the nonnegative integer such that $T^{(3)} \in \left(T^{(4)} + P_1\omega, T^{(4)} + (P_1+1)\omega\right]$. By (4.22), we have

$$\begin{aligned}
&\varepsilon \exp\{M(\varepsilon)\omega\} < y\left(T^{(3)}\right) \\
&< y(T^{(4)} \exp\left\{\int_{T^{(4)}}^{T^{(3)}} \left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2) + x_2(t-\tau_2) + a^*} - d(t)\varepsilon\right) dt\right\} \\
&\quad = \varepsilon \exp\left\{\int_{T^{(4)}}^{T^{(4)}+P_1\omega} + \exp\int_{T^{(4)}+P_1\omega}^{T^{(3)}} \right\}\left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2) + x_2(t-\tau_2) + a^*} - d(t)\varepsilon\right) dt \\
&< \varepsilon \exp\left\{\int_{T^{(4)}+P_1\omega}^{T^{(3)}} \left(-d_3(t) + \frac{a_2(t)x_2(t-\tau_2)}{kx_2^2(t-\tau_2) + x_2(t-\tau_2) + a^*} - d(t)\varepsilon\right) dt\right\} \\
&\quad \leq \exp\{M(\varepsilon)\omega\}.
\end{aligned}$$

Which is a contradiction. This shows that (4.23) holds. By the arbitrariness of $\varepsilon$, it immediately follows that $y(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem 3.1.

**Proof of theorem 3.2.** The proof of theorem 3.2 is similar to that of the proof of Proposition 4.1-4.4 in theorem 4.1, only with a slightly modification, we omit the detail here.

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