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Dynamics of a bounded rational Cournot duopoly model with cooperation

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Abstract. In this paper, a description of a Cournot duopoly model based on a general inverse demand function and a quadratic cost function is investigated. Existence and stability of equilibrium points are investigated analytically and numerically. Cooperation in duopoly is considered with "tit-for tat" strategy.

Keywords: Cournot's duopoly model; Cooperation in duopoly; Existence and Stability of Equilibrium; Simulation modeling.

1Introduction:

An oligopoly is a market form in which a market or an industry is dominated by small number of sellers, called oligopolists. Unlike monopoly, which has no competitor and perfectly & monopolistically competitive firm, which has many competitors, an oligopoly firm faces only few competitors. An oligopoly has greater market power than monopolistic competition and perfect competition, but not as much market power as monopoly. Since there are few participants in this type of market, each oligopolist is aware of actions of the others. Because of the thorough competition, it is often called as 'cut-throat' competition. Many industries in the developed economies are of oligopolistic form of market. This form of market is an emerging phenomenon not only in domestic but in international market as well. Few examples of typical oligopoly markets are: (a) Credit Card (Visa, Master Card, American Express & Discover are competing in the global market), (b) Soft Drink (Pepsi & Coca Cola are competing in Indian Market), (c) Automobile industry [specially family car manufacturing] (Maruti-Suzuki, Tata, Hyundai, Honda, Toyota, General Motors & Ford are competing in the Indian market).A market with just two firms is called a *duopoly*. Obviously, a duopoly is the simplest sort of oligopoly, and many of the concepts and results that we will describe can be extended to the case of an oligopoly withmore than two firms. Duopoly analysis by economists dates back to the 19th century. Some ofthe central concepts of duopoly analysis have to do with strategic behavior, and the analysis of strategic behavior is the heart of the 20th century discipline called *game theory*. Therefore, game theory builds on duopoly theory. It is one of the most important theories that is used to describe and study such competition among competitors statically and dynamically. The dynamic case in which the equilibrium point (Nash equilibrium) is sought and its complex dynamic characteristics are of main interest have been studied in literature [1]-[14]. There are two fundamentally different approaches to duopoly theory. The first assumes that duopolists compete with each other through their choices of *quantity*: each firm decides on the quantity it should produce and sell in the market, contingent on the other firm's quantity. The second assumes that duopolists compete with each other through their choices of *price*: each firm decides on the price it should charge, contingent on the price the other firm is charging. The French mathematician and economist Antoine Augustin Cournot (1801-1877), who wrote about duopoly in 1838, took the first approach. Another French mathematician, Joseph Louis Francois Bertrand (1822-1900), in 1883, developed the second approach.

The Cournot duopolistic game has been studied intensively in literature. Agiza and Elsadany[14] have modeled a Cournot duopolistic game on which one of the competitors is heterogeneous. They have studied the proposed game in details and particularly when the game's fixed point becomes unstable due to bifurcation occurred. In [15], a heterogeneous duopoly game with rational and adaptive competitors was examined. The authors have studied the impact of quadratic cost function on the complex behavior of the game and came up with the conclusion that introducing nonlinearity in cost generalizes the implications of heterogeneous Cournot duopoly with adjusting strategies. Other interesting works related to Cournot duopolistic games can be in literature [16], [17] and [18].

In Cournot duopoly games, Nash equilibrium or Cournot equilibrium is the basic solution in such games and reflects the rationality of the firms within games. Since, firm rationality contradicts with Pareto optimality (in cooperation case), and then Nash equilibrium in duopoly game is not Pareto optimal. In other words, Pareto optimality in such games cannot be achieved by firm interest's maximization. As reported in [19] and [20], theoretical and experimental studies have leaded up to several ways by which the cooperative solution can be obtained. For instance, in the well-known short game of prisoner's dilemma, the Nash equilibrium point is Pareto optimal as cooperation is obtainable. However, for the repeated games, emergence of cooperation among competitors (firms) may be possible to achieve and then cooperation in iterated prisoner's dilemma can be explained. In [21], it has been shown that the conditional cooperative strategy such as the so-called "tit-for-tat" may be used to achieve cooperation among firms in repeated games.

The emergence of cooperation has attracted much of interest for a long time and it would look like even pleonastic to report some of the recent and important papers in this field. In [22], setups based on discrete, continuous and mixed strategy have been proposed in the social dilemma games and their performance on networks populations has been shown. A useful source of information on the evolutionary games on multilayer networks and particularly in the evolution of cooperation is reported in details in [23]. An evolutionary dictator game model is introduced in [24] by which the evolution of altruism and fairness of populations has been studied. In this study, the influence of assignation on heterogeneous populations has been investigated. An important review of the universality of scaling for the dilemma strength in evolutionary games has been reported in [25]. The review has shown that social viscosity or spatial structure causes the existing scaling parameters to fail. In addition to the review has developed new parameters to resolve the paradox of cooperation benefits. Two-layer scale-free networks has been introduced in [26] to show evolution of cooperation. In [27], the authors have demonstrated that the influence of simple strategy-independent form may expand the scope of cooperation on structured populations. For more related works, readers are advised to have a look on some important papers [28] and [29] and a more informational report [30].

Bounded rationality and Puu incomplete information are two different approaches that have been recently used to study monopoly and duopoly markets. Bounded rational players (firms) update their production strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions [11], as all they need to know is how the market will response to small production changes, in order to adjust their production levels by means of a local estimate of the marginal profit. On the other hand, Puu[12] has recently introduced the so-called Puu incomplete information in which, more realistically, a firm does not need to know the local slope of the profit function to choose the quantity to produce in the next time step (see[31] and [10]).Instead, all it needs is its profit and the quantities produced in the past two times.

In this paper, a description of a Cournot duopoly model based on a more general form of downward sloping and concave inverse demand function than the one used in [32] is explored. The two-dimensional map whose iteration gives the time evolution of the output of the two competing firms is defined. Existence and stability of equilibrium points are investigated analytically and numerically. Cooperation in duopoly is considered with "tit-for tat" behavior and control.

2 Duopoly Model:

Assume there are two firms in the market producing a homogenous good. Production decisions occur at discrete times $t = 0,1,2,\dots$ at time t, firm one produces $q_{1,t}$ units of the good; firm 2 produces $q_{2,t}$ units of the good. The total amount produced is $Q_t = q_{1,t} + q_{2,t}$. We consider the inverse demand function $p = f(Q_t) = a - bQ_t^n,\ n \in \mathbb{R}\ ,\ a > 0,\ b > 0.$ We assume firm i has the production *cost function* $C_i(q_{i,t}), i = 1,2,$given by $C_1(q_{1,t}) = c_1 q_{1,t}^2 + c_2 q_{1,t} + c_3$, and $C_2(q_{2,t}) = d_1 q_{2,t}^2 + d_2 q_{2,t} + d_3$, and where $c_j > 0,\ d_j > 0,\ j = 1,2,3$, and $a > \max\{c_2, d_2\}$.

The cost functions $C_i(q_{i,t}), i = 1,2$ are non-negative, convex, and has positive first derivative. The *profit function* for the ith firm, may be written, with q_t denoting the vector of outputs $(q_{1,t}, q_{2,t})$at time t, as :

$$\pi_i(q_t) = q_{i,t} f(Q_t) - C_i(q_{i,t}), \quad i = 1,2$$

or $\pi_i(q_t) = q_{i,t}(a - bQ_t^n) - C_i(q_{i,t}), \quad i = 1,2$

The empirical estimate of the marginal profit is given by

$$\frac{\partial \pi_i(q_t)}{\partial q_{i,t}} = a - bQ_t^n - nbq_{i,t}Q_t^{n-1} - \frac{dC_i(q_{i,t})}{dq_{i,t}}, \quad i = 1,2$$

Then the dynamical system of two players in a Cournot game, which describes the time evolution of the outputs of the two competing firms, is

$$q_{i,t+1} = q_{i,t} + \alpha_i(q_{i,t})\frac{\partial \pi_i(q_t)}{\partial q_{i,t}} \qquad i = 1,2 \qquad (1)$$

Where $\alpha_i(q_{i,t})$ is a positive function that gives the extent of the production variation of the ith firm according to its marginal profit. (1) can then be written as:

$$\begin{cases} q_{1,t+1} = q_{1,t} + \alpha_1(q_{1,t})\left(a - bQ_t^n - nbq_{1,t}Q_t^{n-1} - \dfrac{dC_1(q_{1,t})}{dq_{1,t}}\right) \\ q_{2,t+1} = q_{2,t} + \alpha_2(q_{2,t})\left(a - bQ_t^n - nbq_{2,t}Q_t^{n-1} - \dfrac{dC_2(q_{2,t})}{dq_{2,t}}\right) \end{cases}$$

We take the function$\alpha_i(q_{i,t})$, $i = 1,2$, as linear, that is:$\alpha_i(q_{i,t}) = k_i q_{i,t}$, where k_i is a positive constant, which is called the speed of adjustment of the$i$th firm.

$$\begin{cases} q_{1,t+1} = q_{1,t} + k_1 q_{1,t}\left(a - bQ_t^{n-1}\left((n+1)q_{1,t} + q_{2,t}\right) - 2c_1 q_{1,t} - c_2\right) \\ q_{2,t+1} = q_{2,t} + k_2 q_{2,t}\left(a - bQ_t^{n-1}\left((n+1)q_{2,t} + q_{1,t}\right) - 2d_1 q_{2,t} - d_2\right) \end{cases} \qquad (2)$$

2.1 Existence of equilibrium points and local stability:

Equilibrium points of the nonlinear dynamical system (2) are given by the solutions to the following nonlinear dynamical system, obtained by letting

$q_{i,t+1} = q_{i,t}$, $i = 1,2$in (2)

$$\begin{cases} k_1 q_{1,t}\left(a - bQ_t^{n-1}\left((n+1)q_{1,t} + q_{2,t}\right) - 2c_1 q_{1,t} - c_2\right) = 0 \\ k_2 q_{2,t}\left(a - bQ_t^{n-1}\left((n+1)q_{2,t} + q_{1,t}\right) - 2d_1 q_{2,t} - d_2\right) = 0 \end{cases}$$

There are four equilibrium points given by:

$E_0 = (0,0)$, $E_1 = (\bar{q}_1, 0)$, $E_2 = (0, \bar{q}_2)$, $E_3 = (q_1^*, q_2^*)$,

The equilibrium points E_0, E_1, $E_2$are called the boundary equilibrium points, and the equilibrium point E_3 is called Nash equilibrium. Now, we show the existence of $\bar{q}_1$and$\bar{q}_2$. Letting $q_1 \neq 0$, and $q_2 = 0$ in system (2), we get the following equation:

$$b(n+1)q_1^n + 2c_1 q_1 + c_2 - a = 0$$

Define the function$\varphi(q_1) = b(n+1)q_1^n + 2c_1q_1 + c_2 - a$. It can be shown easily that $\varphi(q_1)$ is a strictly increasing convex function, and that$\varphi(0) = c_2 - a < 0$,and $\varphi\left(\frac{a-c_2}{2c_1}\right) = b(n+1)\left[\frac{a-c_2}{2c_1}\right]^n > 0$. From these properties of the function $\varphi(q_1)$, we conclude, from Intermediate value theorem, that there exists a unique value of q_1 , call it $\overline{q}_1$, such that $\varphi(\overline{q}_1) = 0$. Similarly, the function$\gamma(q_2) := b(n+1)q_2^n + 2d_1q_2 + d_2 - a$ has a unique value$\overline{q}_2$ such that $\gamma(\overline{q}_2) = 0$. Byapplying Theorem 1 [33], we conclude the existence of Nash equilibriumE_3.

Numerical experiments are simulated to investigate the stability of the two-dimensional system (2). In what follows, we take for simplicity $k_i = k,\ i = 1,2$.The dynamical behavior of map (2) is carried out by fixing the model parameters as follows: a = 2.0, b =0.3, c1=0.11, c2=0.15, d1=.9, d2=0.25, n=3. Fig. 1 shows the bifurcation diagram of the dynamic system with initial conditionq1_0 =0.1, and q2_0 = 0.1.The equilibrium point is$q_1 = 0.836$ and $q_2 = 0.383$ and is locally stable when k lies in the interval [0.0, 0.412].For ≥ 0.412 , there is transition to chaos via a sequence of period-doubling bifurcations. In Fig.2 , the Maximum Lyapunov Exponent (MLE) corresponding to the bifurcation diagram given in Fig. 1 is plotted. Positive values of maximum Lyapunov exponents show that the solution have chaotic behavior. Negative maximum Lyapunov exponents corresponds to a stable coexistence of the system. Fig. 3 exhibits the chaotic attractor of the map (2) for the same parameters. One of the properties of a dynamical system to be chaotic is its sensitivity to initial conditions. In Fig. 4 the sensitivity of the system (when the system loose stability) is carried out with the previous values of the parameters. The figure presents two orbits of the first firm when $q_{1,1} = 0.1$ and with some deviation $q_{1,1} = 0.1 + 0.0001$. At the beginning, the two orbits are identical, then the distinction between them builds up rapidly.

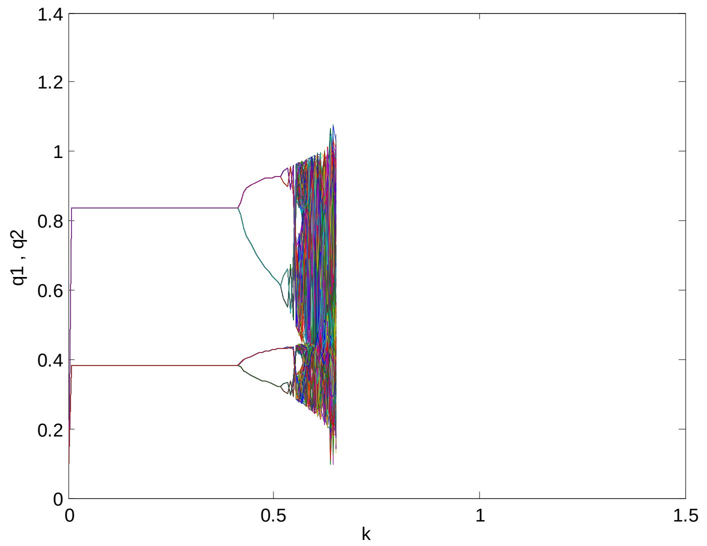

Fig. 1 Bifurcation Diagram of q1 and q2 of system (2) versus k

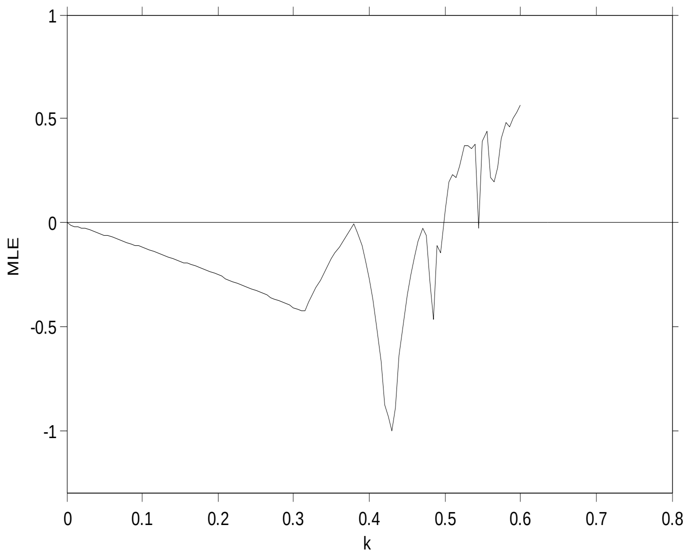

Fig. 2 MaximumLyapounov Exponent of system (2) versus k

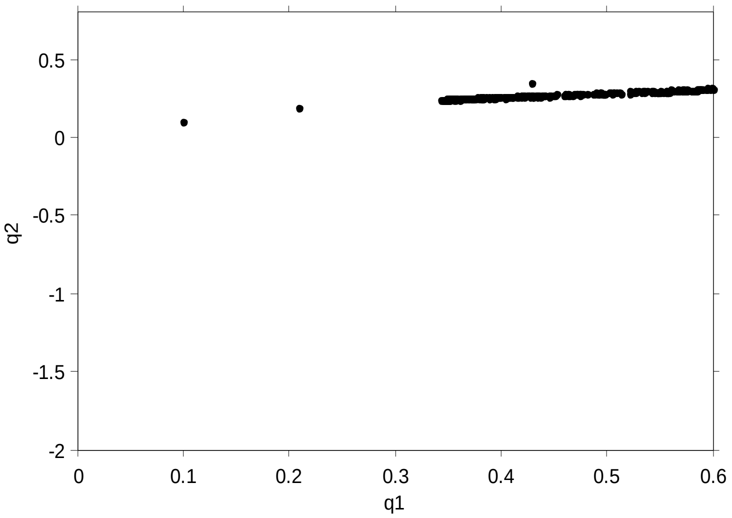

Fig. 3 Chaotic Attractor of system (2)

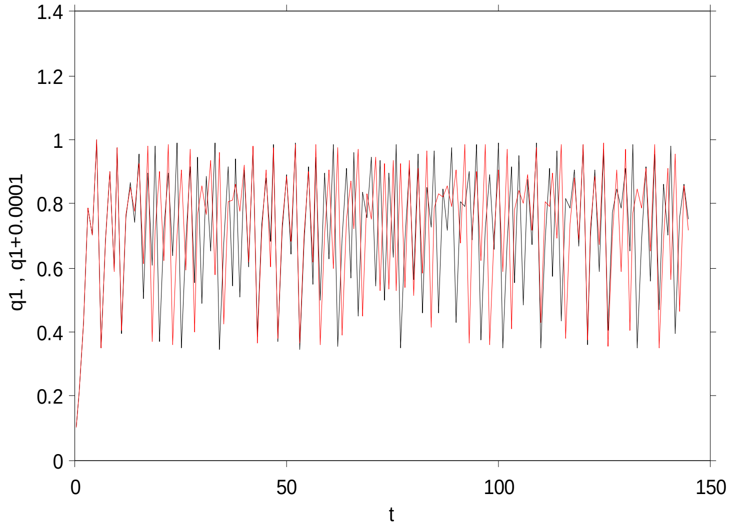

Fig. 4 Chaos Sensitivity of system (2)

2.2 Case of linear production cost function :

In case, the production cost functions are linear, that is $C(q_{i,t}) = c_i q_{i,t} + d_i$, where $c_1 = c_2 = c > 0$,and $a >$ c, system (2) becomes

$$\begin{cases} q_{1,t+1} = q_{1,t} + k_1 q_{1,t}\left(a - bQ_t^{n-1}\left((n+1)q_{1,t} + q_{2,t}\right) - c\right) \\ q_{2,t+1} = q_{2,t} + k_2 q_{2,t}\left(a - bQ_t^{n-1}\left((n+1)q_{2,t} + q_{1,t}\right) - c\right) \end{cases} \qquad (3)$$

The equilibrium points of system (3) are obtained explicitly and given by

$$E_0 = (0,0), \quad E_1 = \left(\left(\frac{a-c}{(n+1)b}\right)^{1/n}, 0\right),$$

$$E_2 = \left(0, \left(\frac{a-c}{(n+1)b}\right)^{1/n}\right), \quad E_3 = \left(\left(\frac{a-c}{2^{n-1}(n+2)b}\right)^{1/n}, \left(\frac{a-c}{2^{n-1}(n+2)b}\right)^{1/n}\right),$$ where E_3 is the

only Nash equilibrium.

The local stability of the fixed points of the two dimensional system (3) depends on the eigenvalues of the Jacobian matrix of (3).

The Jacobian matrix at the point $(q_1, q_2)$is given by $J(q_1, q_2) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$, where:

$$J_{11} = 1 + k_1[a - c - bQ^{n-2}([(n+1)q_1 + q_2]^2 + nq_1q_2)]$$

$$J_{22} = 1 + k_2[a - c - bQ^{n-2}([(n+1)q_2 + q_1]^2 + nq_1q_2)]$$

$$J_{12} = -nk_1 bQ^{n-2}(nq_1^2 + q_1q_2)$$

$J_{21} = -nk_2bQ^{n-2}(nq_2^2 + q_1q_2)$

At the equilibrium point $E_0 = (0,0)$, we have

$$J(0,0) = \begin{bmatrix} 1 + k_1(a-c) & 0 \\ 0 & 1 + k_1(a-c) \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 1 + k_1(a-c) > 1$, and$\lambda_2 = 1 + k_2(a-c) > 1$. Hence the equilibrium point E_0 is unstable. At the equilibrium point $E_1 = \left(\left(\frac{a-c}{(n+1)b}\right)^{1/n}, 0\right)$, we have

$$J(E_1) = \begin{bmatrix} 1 - k_1n(a-c) & -n^2k_1(a-c)/n+1 \\ 0 & 1 + (n^2k_2(a-c)/n+1) \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 1 - k_1n(a-c) < 1$, and

$\lambda_2 = 1 + n^2k_2(a-c)/n + 1 > 1$. Hence, the equilibrium point E_1 is unstable. At the equilibrium point $E_2 = \left(0, \left(\frac{a-c}{(n+1)b}\right)^{1/n}\right)$, we have

$$J(E_2) = \begin{bmatrix} 1 + (nk_1(a-c)/(n+1)) & 0 \\ -n^2k_2(a-c)/n+1 & 1 - k_2n(a-c) \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 1 + [nk_1(a-c)/(n+1)] > 1$, and$\lambda_2 = 1 - k_2n(a-c) < 1$. Hence, the equilibrium point E_2 is unstable.

At the Nash equilibrium point $E_3 = \left(\left(\frac{a-c}{2^{n-1}(n+2)b}\right)^{1/n}, \left(\frac{a-c}{2^{n-1}(n+2)b}\right)^{1/n}\right)$, we have

$$J(E_3) = \begin{bmatrix} 1 - (k_1n(n+3)(a-c)/2(n+2)) & -k_1n(n+1)(a-c)/2(n+2) \\ -k_2n(n+1)(a-c)/2(n+2) & 1 - (k_2n(n+3)(a-c)/2(n+2)) \end{bmatrix}$$

The characteristic equation is$p(\lambda) = \lambda^2 - Tr\big(J(E_3)\big) + Det(J(E_3) = 0$, where

$Tr\big(J(E_3)\big)$is the sum of the main diagonal entries of the Jacobian matrix$J(E_3)$and $Det(J(E_3)$ is the determinant of the Jacobian matrix $J(E_3)$, that is

$Tr\big(J(E_3)\big) = 2 - ((k_1 + k_2)(a-c)n(n+3)/2(n+2))$, and

$Det(J(E_3)) = 1 - (k_1 + k_2)(a-c)n(n+3)/2(n+2) + k_1k_2(a-c)^2\, n^2/(n+2)$

According to Jury's conditions, the Nash equilibrium point E_3 is locally asymptotically

stable if $\left|Tr\big(J(E_3)\big)\right| < 1 + Det(J(E_3) < 2$ or equivalently:

i) $Det(J(E_3) < 1$

ii) $1 - Tr(J(E_3)) + Det(J(E_3) > 0$

iii) $1 + Tr(J(E_3)) + Det(J(E_3) > 0$

Condition ii) is obviously satisfied. Conditions i) and iii) can be expressed as

$$k_1 + k_2 > k_1 k_2 \frac{2n(a-c)}{n+3} \text{and} (k_1 + k_2) \frac{n(n+3)(a-c)}{n+2} < 4 + k_1 k_2 \frac{n^2(a-c)^2}{n+2}.$$

Therefore E_3 is locally asymptotically stable if the above conditions are satisfied.

Numerical simulation are carried to show that the stability and period doubling bifurcation to chaos for system (3). As in the previous model, for simplicity, $k_i = k,\ i = 1,2$. The parameters are taken to be a =2.0, b=0.3, c=0.15, n=3. The Nash equilibrium at these values isE_3 =(0.6756 , 0.6756). The bifurcation diagram in Fig. 5 shows that the trajectories converge to the equilibrium (0.6756, 0.6756) for $k < 0.36$ and for$k > 0.36$, the Nash equilibrium becomes unstable, period doubling bifurcation appears at $k = 0.36$ and finally chaotic behavior occurs. In Fig. 6, the maximal Lyapunov exponent is plotted. A positive MLE is usually taken as an indication that the system is chaotic. In the range $0 < k < 0.36$, the maximum Lyapunov exponents are negative, corresponding to a stable coexistence of the system. While in range $0.36 < k < 0.6$, most Lyapunov exponents are positive, and few are negative. This means that there exist stable fixed points or periodic windows in the chaotic band. Fig. 7 exhibits the chaotic attractor of the map (3) for the same parameters and$k$ is fixed to 0.58. In Fig. 8, the sensitivity of the system when the system becomes chaotic is carried out with the previous parameters and $k = 0.58$. The figure presents two orbits of the first firm when $q_{1,1} = 0.1$ and with some deviation $q_{1,1} = 0.1 + 0.0001$. At the beginning the time series are indistinguishable, but after a number of iterations, the difference between them builds up rapidly.

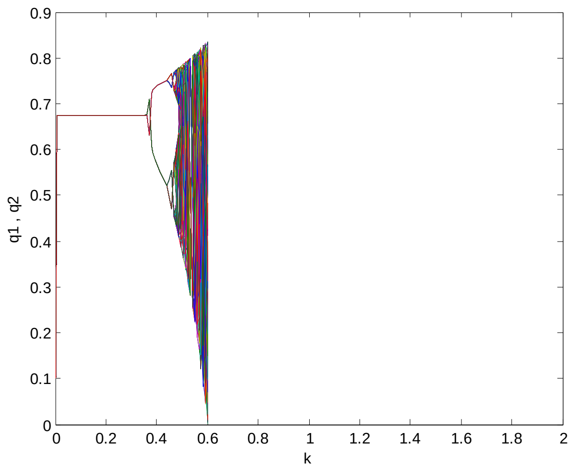

Fig. 5 Bifurcation Diagram of q1 and q2 of system (3) versus k

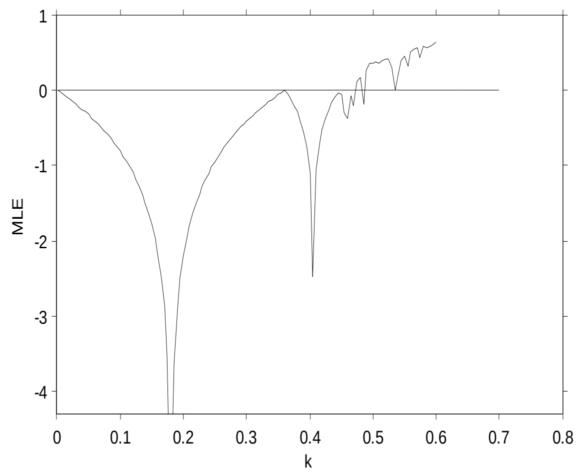

Fig. 6 Maximum Lyapunov Exponent of system (3) versus k

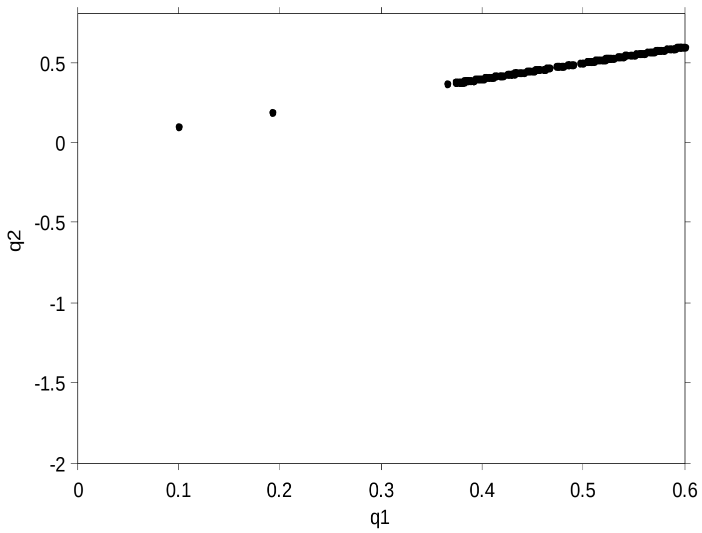

Fig. 7 Chaotic Attractor of system (3)

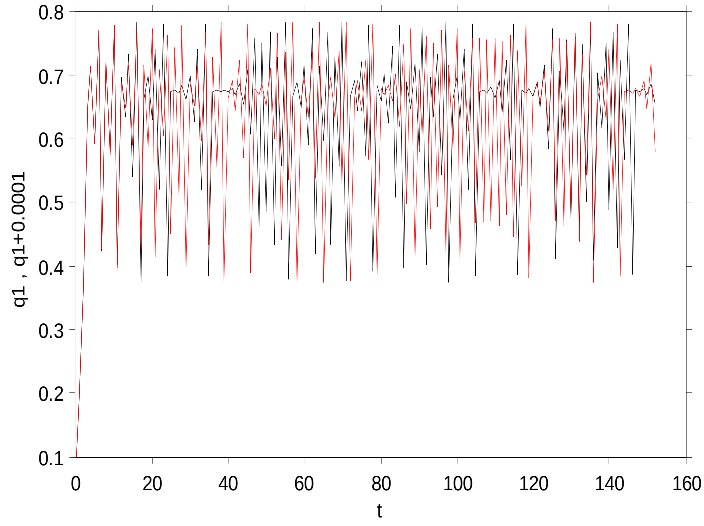

Fig. 8 Chaos Sensitivity of system (3)

3 Cooperative Duopoly Model :

In this section, the same assumptions made on the previous model (2) are considered. In particular, firm i has a quadratic production *cost function* $C_i(q_{i,t}) = c_i q_{i,t}^2 + d_i q_{i,t} + e_i$, $i = 1,2$,and where$c_j \geq 0$, $d_j \geq 0, e_j \geq 0$, $j = 1,2,3$, and $a > \max\{c_1, c_2\}$.

The *profit function* for the i^{th} ($i = 1,2$) firm is given by

$$\pi_i(q_{1,t}, q_{2,t}) = q_{i,t}(a - bQ_t^n) - C_i(q_{i,t})$$
$$= q_{i,t}(a - bQ_t^n) - (\, c_i q_{i,t}^2 + d_i q_{i,t} + e_i) \,, \qquad i = 1,2.$$

Where $q_t = (q_{1,t}, q_{2,t})$and $Q_t = q_{1,t} + q_{2,t}$.

Hence, the empirical estimate of the marginal profit is:

$$\frac{\partial \pi_i(q_{1,t}, q_{2,t})}{\partial q_{i,t}} = a - bQ_t^n - nbq_{i,t}Q_t^{n-1} - 2c_i q_{i,t} - d_i, \quad i = 1,2,$$

Cooperation under incomplete information is based on the assumption that the firms compare their own profits with the cooperative profit. The cooperative profit means the profit, which is solved by maximizing the sum of all firms' profit. That is:

$$max\ \varphi(q_{1,t}, q_{2,t}) = \pi_1(q_{1,t}, q_{2,t}) + \pi_2(q_{1,t}, q_{2,t})$$

Next, it is proved that the function $\varphi(q_{1,t}, q_{2,t})$ is strictly concave, and therefore has a unique global maximum. This global maximum is obtained by applying first order optimality conditions to the above unconstrained problem. That is, by solving the equation $\nabla\varphi(q_{1,t}, q_{2,t}) = 0$ which ca be written as:

$$\nabla\varphi\left(q_{1,t},q_{2,t}\right)=\begin{bmatrix}\frac{\partial\varphi\left(q_{1,t},q_{2,t}\right)}{\partial q_{1,t}}\\ \frac{\partial\varphi\left(q_{1,t},q_{2,t}\right)}{\partial q_{2,t}}\end{bmatrix}=\begin{bmatrix}a-(n+1)bQ_t^n-2c_1q_{1,,t}-d_1\\ a-(n+1)bQ_t^n-2c_2q_{2,t}-d_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad (4)$$

Remark 1 If$(q_{1,t},q_{2,t})$ satisfies (4), then necessarily it satisfies the following relation :
$q_{1,t}=q_{2,t}+\frac{d_2-d_1}{2(c_1-c_{2)}}$ and if$d_1=d_2$, then $q_{1,t}=q_{2,t}$.

The Hessian matrix of $\varphi(q_{1,t},q_{2,t})$is given by:

Clearly, $H\left(q_{1,t},q_{2,t}\right)$ is negative definite, and therefore the function $\varphi(q_{1,t},q_{2,t})$ is strictly concave, which implies that it has a unique global maximum. Let us denote it by (q_1^c,q_2^c).For the firms, (q_1^c,q_2^c) represents the optimal cooperative output, and $\pi_c=\frac{\pi_1(q_1^c,q_2^c)+\pi_2(q_1^c,q_2^c)}{2}$, the cooperative profit.

For achieving the cooperation between the two firms, the tit-for-tat strategy is used. With this strategy, every firm is doing what its opponent has done in the previous move. The tit-for-tat strategy is the best behavior allowing the achievement of cooperation in repeated games [34]. Its characteristic is that every player consists in doing what the opponent did in previous move. This is an incomplete information scenario. However, the only thing each firm knows are the output and the profit. In this situation, each firm i, will compare its profit $\pi_{i,t}{}^{(*)}, i=1,2,$,with the cooperative profit π_c that is Pareto optimal . If $\pi_{i,t}>\pi_c$ then this means that each firm will probably reduce its output to keep the cooperation between them. On the other hand, if $\pi_{i,t}<\pi_c$, then this condition indicates that cooperation cannot be realized. Based on this argument, the following dynamic map is built:

$$\begin{cases}q_{1,t+1}=q_{1,t}+\alpha_1\left(\pi_c-\pi_{1,t}\right)\\ q_{2,t+1}=q_{2,t}+\alpha_2\left(\pi_c-\pi_{2,t}\right)\end{cases}$$

$(*)\ \pi_{i,t}=\pi_i\left(q_{1,t},q_{2,t}\right), i=1,2.$

Where $\alpha_i>0\ (i=1,2)$ is an adjustment parameter.For simplicity, $\alpha_i=k,\ i=1,2.$

which can be written as

$$\begin{cases}q_{1,t+1}=q_{1,t}+k\left(\pi_c-q_{1,t}(a-bQ_t^n)+c_1q_{1,t}^2+d_1q_{1,t}+e_1\right)\\ q_{2,t+1}=q_{2,t}+k\left(\pi_c-q_{2,t}(a-bQ_t^n)+c_2q_{2,t}^2+d_2q_{2,t}+e_2\right)\end{cases}(5)$$

In [19], [34], [35], the authors has used a special case of cost functions than the ones used in our model. Namely, they consider linear cost functions, that is: $c_1=c_2=0,\ d_1=d_2=d>0, e_1=$

$e_2 = 0$. In this special case, the cooperative output (q_1^c, q_2^c), and the cooperative profit $\pi_c$are derived easily. Also, it happens that, in this case, the unique equilibrium point of the dynamical system (5) is exactly the optimal cooperative output (q_1^c, q_2^c). But, in general, this does not hold for the quadratic case.

Due to the difficulty of determining the equilibrium points of the discrete time dynamical system (5) in explicit closed forms, numerical simulation will be performed to investigate the stability of the system. After running a number of times the simulation on various parameters values of this model and for a given value of the cooperative profit π_c, it appears that, there are two main parameters that have the most impact on the stability of the dynamical system. Namely, the adjustment parameter k and the exponent n present in the demand function. Two values of n(0.5and 0.01) are considered to illustrate the effect on the behavior of the model. The parameters are taken to be a = 4.0, b = 0.3, c1 = 0.11, c2 = 0.15, d1 =.9, d2 = 0.25, and n =0.5. The equilibrium at these values isq_1 =4.386and$q_2 = 2.61$. The bifurcation diagram in Fig. 9 a. shows that the trajectory of the solutionq_1 converges to the equilibrium 4.386for $k < 0.784$ and hence Pareto optimality may be reached. For $k > 0.784$, the equilibrium becomes unstable, period doubling bifurcation appears at $k = 0.784$,and finally, chaotic behavior occurs. Similarly, the trajectory of the solutionq_2 converges to the equilibrium 2.61 for $k < 0.776$. For $k > 0.776$, the equilibrium becomes unstable, period doubling bifurcation appears at $k = 0.776$ and finally chaotic behavior occurs. Fig. 10 a shows the Maximum Lyapunov Exponent corresponding to Fig. 9aNotice that positive MLE indicates chaotic conduct. Fig. 11 a exhibits the chaotic attractor of the map (5) for the same parameters and$k$ is fixed to 1.11 in the chaotic range. Fig. 12 a gives the sensitivity of the system in chaotic state with $k = 1.12$. The figure presents two orbits of the first firm when $q_{1,1} = 0.11$ and with some deviation $q_{1,1} = 0.11 + 0.0001$. At the beginning, the time series are indistinguishable, but after a number of iterations, the difference between them builds up rapidly.

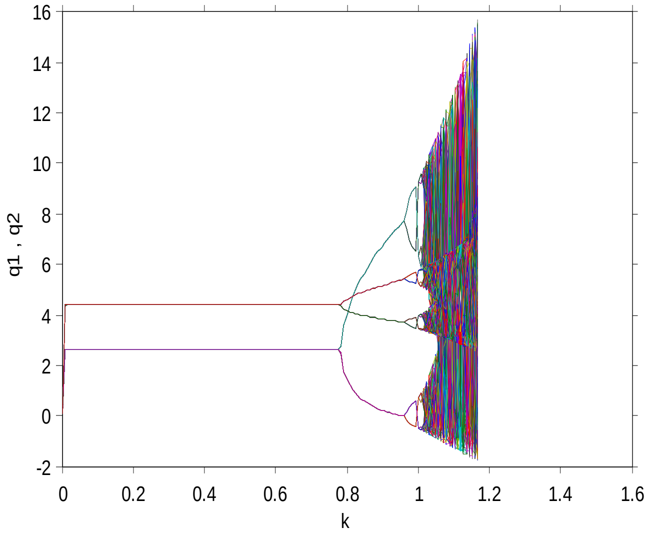

Fig. 9a Bifurcation Diagram of q1 and q2 of system (5) versus k with n=0.5

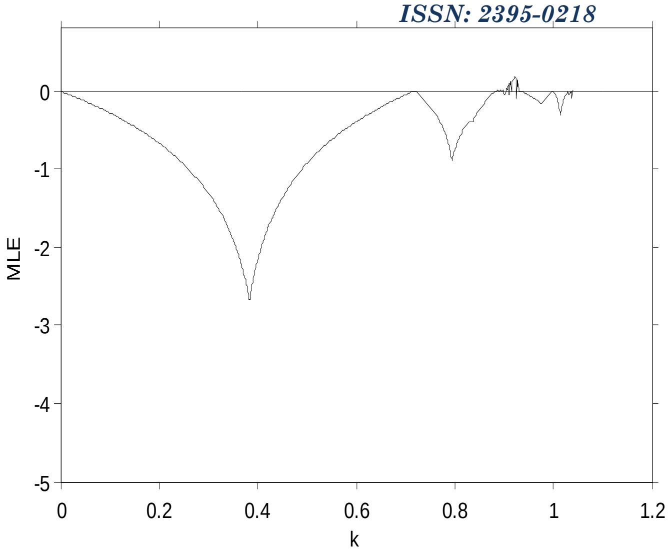

Fig. 10a MaximumLyapounov Exponent of system (5) versus k with n=0.5

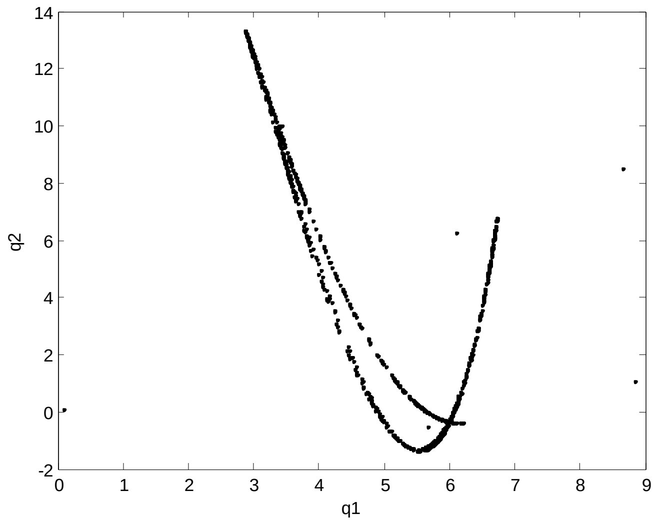

Fig. 11a Chaotic Attractor of system (5) with n=0.5

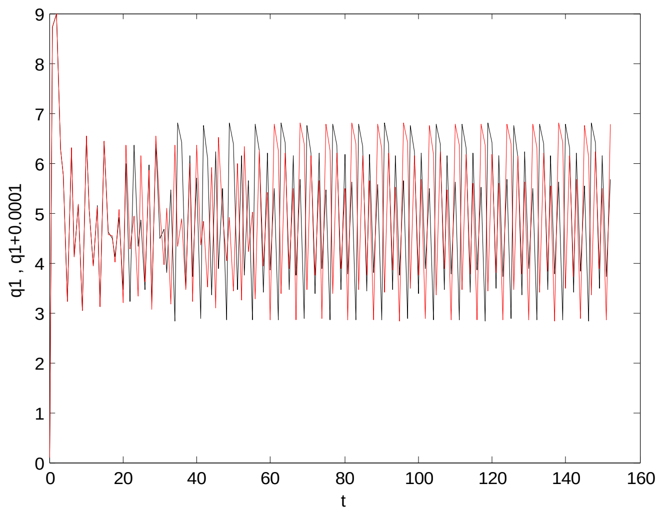

Fig. 12a Chaos Sensitivity of system (5) with n=0.5

For the case n =0.01, and the same values of the other parameters. That is:a = 4.0, b = 0.3, c1 = 0.11, c2 = 0.15, d1 =.9, d2 = 0.25. Comparing the figures in case n =0.01 with the ones with n = 0.5, they indicate that when n =0.01, the chaotic pattern has beenreduced drastically. The bifurcation diagram in Fig. 9b shows that the trajectory of the solutionq_1 is stable and converges to the equilibrium 3.288. On the other hand, the trajectory of the solutionq_2 converges to the equilibrium 2.214 for$k < 0.608$ and for $k > 0.608$, period doubling bifurcation appears at $k = 0.608$ and lastly, chaotic behavior occurs and hence Pareto optimality cannot be achieved.

Fig. 10b shows the Maximum Lyapunov Exponent corresponding to Fig. 9b Here MLE indicates very less chaotic

conduct.Fig. 11b exhibits the chaotic attractor of the map (5) for the same parameters and k is fixed to 1.12. Fig. 12b gives the sensitivity of the system in chaotic state with $k = 0.78$. The figure presents two orbits of the first firm when $q_{1,1} = 0.11$ and with some deviation $q_{1,1} = 0.11 + 0.0001$. The figure shows that the difference between the two orbits is highly less sensitive in comparison with Fig. 12a (n = 0.01). This is explained by the fact that the chaotic region in Fig. 9a is much important than the one in Fig. 9b.

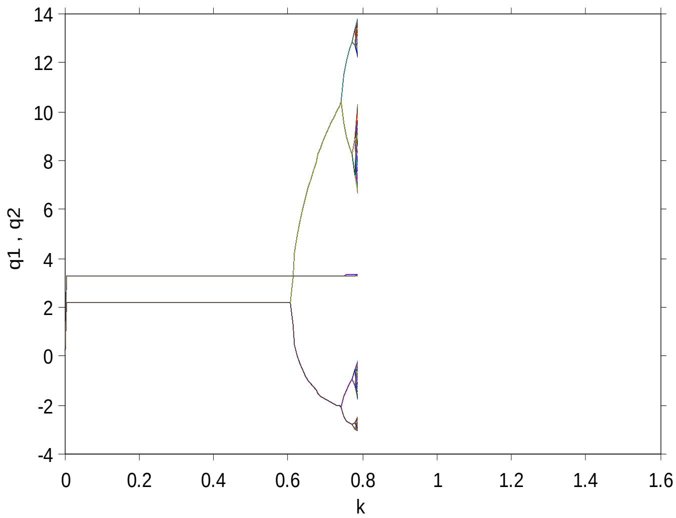

Fig. 9b Bifurcation Diagram of q1 and q2 of system (5) versus k with n=0.01

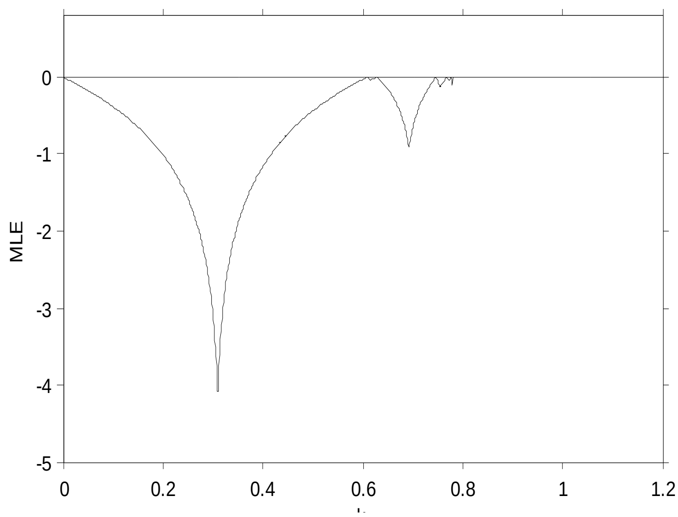

Fig. 10b Maximum Lyapounov Exponent of system (5) versus k with n=0.01

5The conclusion :

In this paper, based on a general nonlinear inverse demand function and a quadratic cost function, three duopolostic Cournot models have been investigated. For the first two models, existence of equilibrium points and local stability has been studied. Numerical simulations on the dynamical systems has been performed to show bifurcations diagrams, stability regions and chaos. The third model is a cooperative Cournot duopoly game under the tit-for-tat strategy with incomplete information scenario. The analysis shows that Pareto optimality cannot be certain.

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