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Oscillation of θ -methods for the Lasota-Wazewska model

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Abstract: The aim of this paper is to discuss the oscillation of numerical solutions for the Lasota-Wazewska model. Using two θ -methods (the linear θ -method and the one-leg θ -method), some conditions under which the numerical solutions oscillate are obtained for different range of parameter θ . Furthermore, it is shown that every non-oscillatory numerical solution tends to the fixed point of the original continuous equation. Numerical examples are given.

Keywords: Lasota-Wazewska model; θ -methods; Oscillation; Non-oscillation.

1 Introduction

Nowadays, there are much scientific activity concerning the oscillatory behavior of difference equations[1], dynamic equations[2], hyperbolic equations[3], fractional differential equations[4], hybrid systems[5] and differential equations with piecewise continuous arguments (DEPCA)[6]. Among these studies, oscillations of solutions of delay differential equations (DDEs) have also been the subject of many recent investigations [7, 8]. The intensive interest in this subject is motivated by the fact that it has many useful applications in some mathematical models, such as biology, ecology, spread of some infectious diseases in humans and so on.

It was worth noting that much research has been focused on the oscillation properties of the numerical solutions for DDEs. In [9, 10], numerical oscillation for a retarded DEPCA was considered, respectively. Wang et al. [11] studied oscillation of alternately advanced and retarded DEPCA from numerical aspect. However, for nonlinear DDEs, until now very few results dealing with the numerical oscillation have been found except for [12]. Different from [12], in the present paper, we will study the oscillation of numerical solution for the Lasota-Wazewska model.

Consider the following equation

$$N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}, \quad t \geq 0 \tag{1}$$

with initial condition

$$N(t) = \psi(t), \quad -\tau \leq t \leq 0, \tag{2}$$

where $\psi \in C([-\tau, 0], (0, \infty)), \quad \psi(0) > 0$. Denote N^* is the positive fixed point of (1), then $N^* = pe^{-\gamma N^*} / \mu$. (1) has been used by Wazewska-Czyzewska and Lasota [13] as a model for the survival of red blood cells in an animal. Here $N(t)$ denotes the number of red blood cells at time t, the parameter $\mu > 0$ is the probability of death of a red blood cell, p and γ are positive constants related to the production of red blood cells per unit of time, and τ is the time required to produce a red blood cell.

There are many papers concern different properties of (1). The oscillation and attractivity have been extensively studied in [14]. The existence of periodic solutions has been given in [15]. Mallet-Paret and Nussbaum [16] presented a deep analysis of a class of nonlinear equations with one delay which includes (1). In addition, the generalized Lasota-Wazewska model [17], the impulsive Lasota-Wazewska model [18] and the discrete Lasota-Wazewska model [19] have been comprehensive investigated, respectively. Nevertheless, up to now, few results on the properties of numerical solutions for (1) were established. In the present paper, we will investigate numerical oscillation for (1). We also analyze the asymptotic behavior of non-oscillatory numerical solutions.

Next, we shall address some statements which are useful in presenting the main results of the paper.

Theorem 1 ([14]) Consider the difference equation

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0, \tag{3}$$

assume that $k, l \in \mathbf{N}$ and $q_j \in \mathbf{R}$ for $j = -k, \cdots, l$, then the following statements are equivalent:

(a) Every solution of (3) oscillates;

(b) The characteristic equation $\lambda - 1 + \sum_{j=-k}^{l} q_j \lambda^j = 0$ has no positive roots.

Theorem 2 ([14]) Consider the difference equation

$$a_{n+1} - a_n + \omega a_{n-k} = 0, \quad n = 0, 1, 2, \cdots \tag{4}$$

where $\omega \in \mathbf{R}$ and $k \in \mathbf{Z}$, then every solution of (4) oscillates if and only if one of the following conditions holds:

(a) $k = -1$ and $\omega \le -1$;

(b) $k = 0$ and $\omega \ge 1$;

(c) $k \in \{\cdots, -3, -2\} \cup \{1, 2, \cdots\}$ and $\omega (k+1)^{k+1} / k^k > 1$.

From the linearized oscillation theory in [14], we have the following theorem.

Theorem 3 The solution of (1) and (2) oscillates about N^* if and only if

$$\mu \gamma \tau N^* e^{\mu \tau} > \frac{1}{e}. \tag{5}$$

2 Oscillation of Numerical Solutions

In order to reduce the complexity, we introduce a transformation $N(t) = N^* + x(t)/\gamma$, which can change (1) into the form

$$x'(t) + \mu f_1(x(t)) + \mu \gamma N^* f_2(x(t-\tau)) = 0, \tag{6}$$

where $f_1(u) = u$, $f_2(u) = 1 - e^{-u}$. Clearly, $N(t)$ oscillates about N^* if and only if $x(t)$ oscillates about zero.

Let $h = \tau / m$ be a given step-size with $m \ge 1$, application of the linear θ-method and the one-leg θ-method to (6) gives

$$x_{n+1} = x_n - h\theta\mu f_1(x_{n+1}) - h\theta\mu\gamma N^* f_2(x_{n+1-m}) - h(1-\theta)\mu f_1(x_n) - h(1-\theta)\mu\gamma N^* f_2(x_{n-m}), \tag{7}$$

where $0 \le \theta \le 1$, x_{n+1} and x_{n+1-m} are approximations to $x(t)$ and $x(t-\tau)$ of (6) at t_{n+1}, respectively.

Set $x_n = \gamma(N_n - N^*)$, then we have

$$N_{n+1} = \frac{1-h(1-\theta)\mu}{1+h\theta\mu} N_n + \frac{h\theta\mu N^*}{1+h\theta\mu} \exp\left(-\gamma(N_{n+1-m} - N^*)\right) + \frac{h(1-\theta)\mu N^*}{1+h\theta\mu} \exp\left(-\gamma(N_{n-m} - N^*)\right). \tag{8}$$

It is known that N_n oscillates about N^* if and only if x_n is oscillatory. In order to study oscillation of (8), we only need to consider the oscillation of (7). The following conditions which are taken from [14] will be used next.

$$uf_i(u) > 0 \text{ for } u \neq 0 \text{ and } \lim_{u\to 0} \frac{f_i(u)}{u} = 1, \quad i = 1, 2. \tag{9}$$

The linearized form of (7) is

$$x_{n+1} = x_n - h\theta\mu x_{n+1} - h\theta\mu\gamma N^* x_{n+1-m} - h(1-\theta)\mu x_n - h(1-\theta)\mu\gamma N^* x_{n-m}, \tag{10}$$

equivalently

$$x_{n+1} = \frac{1-h(1-\theta)\mu}{1+h\theta\mu} x_n - \frac{h\theta\mu\gamma N^*}{1+h\theta\mu} x_{n+1-m} - \frac{h(1-\theta)\mu\gamma N^*}{1+h\theta\mu} x_{n-m}. \tag{11}$$

It follows from [14] that (7) oscillates if (11) oscillates under the condition (9).

In the following, we will study whether the θ-methods preserve the oscillation of (1). That is, when Theorem 3 holds, we will investigate the conditions under which (8) is oscillatory.

Let $x_n = \lambda^n x_0$ in (10), by simple computation we have the characteristic equation of (10).

Lemma 1 The characteristic equation of (10) is given by

$$\lambda = R\left(-h\left(\mu + \mu\gamma N^* \lambda^{-m}\right)\right), \tag{12}$$

where $R(x) = {(1+(1-\theta)x)}\big/{(1-\theta x)}$ is the stability function of the θ-methods.

Lemma 2 If condition (5) holds, then (12) has no positive roots for $\theta \in [0, 1/2]$.

Proof: Set $B(\lambda) = \lambda - R\left(-h\left(\mu + \mu\gamma N^* \lambda^{-m}\right)\right)$. Then by Lemma 3 in [20] we have

$$R\left(-h\left(\mu + \mu\gamma N^* \lambda^{-m}\right)\right) \le \exp\left(-h\left(\mu + \mu\gamma N^* \lambda^{-m}\right)\right). \tag{13}$$

We need to prove $E(\lambda) = \lambda - \exp\left(-h\left(\mu + \mu\gamma N^* \lambda^{-m}\right)\right) > 0$ for $\lambda > 0$. If it is not the case, there exists $\lambda_0 > 0$ such that $E(\lambda_0) \le 0$, then $\lambda_0 \le \exp\left(-h\left(\mu + \mu\gamma N^* \lambda_0^{-m}\right)\right)$, further

$$\lambda_0^m \le \exp\left(-\tau(\mu + \mu\gamma N^* \lambda_0^{-m})\right), \tag{14}$$

so

$$\mu\gamma N^* e^{\mu\tau}\tau e \le \mu\gamma N^*\tau\lambda_0^{-m}\exp\left(1-\mu\gamma N^*\tau\lambda_0^{-m}\right).$$

(I) If $1-\mu\gamma N^*\tau\lambda_0^{-m}=0$, then $\mu\gamma N^*\tau e^{\mu\tau}e \le 1$, we get the contradiction with the condition (5).

(II) If $1-\mu\gamma N^*\tau\lambda_0^{-m}\neq 0$, it is seen from the fact "$e^x < 1/(1-x)$ for $x<1$ and $x\neq 0$" that

$$\exp\left(1-\mu\gamma N^*\tau\lambda_0^{-m}\right) < \frac{1}{1-\left(1-\mu\gamma N^*\tau\lambda_0^{-m}\right)} = \frac{1}{\mu\gamma N^*\tau\lambda_0^{-m}},$$

which implies

$$\mu\gamma N^*\tau\lambda_0^{-m}\exp\left(1-\mu\gamma N^*\tau\lambda_0^{-m}\right) < 1,$$

that is

$$\mu\gamma N^*\tau e^{\mu\tau} e < 1,$$

which is also a contradiction to (5). Thus

$$B(\lambda) = \lambda - R\left(-h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)\right) \ge \lambda - \exp\left(-h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)\right) = E(\lambda) > 0,$$

which implies that (12) has no positive roots. The proof is completed.

Without loss of generality, we assume $m>1$ in the case of $\theta\in(1/2,1]$.

Lemma 3 If the condition (5) holds and $\theta\in(1/2,1]$, then the characteristic equation (12) has no positive roots for $h<h_0$, where

$$h_0 = \begin{cases} \infty, & if \quad \mu\gamma N^*\tau \ge 1, \\ \dfrac{\tau(1+\mu\tau+\ln\mu\gamma N^*\tau)}{1+\mu\tau-\mu\tau\ln\mu\gamma N^*\tau}, & if \quad \mu\gamma N^*\tau < 1. \end{cases} \tag{15}$$

Proof: When $\lambda>0$, we know that the function $R\left(-h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)\right)$ is increasing about θ, then for $\lambda>0$ and $\theta\in(1/2,1]$

$$R\left(-h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)\right) = \frac{1-h(1-\theta)\left(\mu+\mu\gamma N^*\lambda^{-m}\right)}{1+h\theta\left(\mu+\mu\gamma N^*\lambda^{-m}\right)} \le \frac{1}{1+h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)}.$$

Now we will prove that the inequality

$$\lambda - \frac{1}{1+h\left(\mu+\mu\gamma N^*\lambda^{-m}\right)} > 0 \tag{16}$$

holds under certain conditions. In view of (16), we have

$$\lambda - \frac{1}{1 + h\left(\mu + \mu\gamma N^{*}\lambda^{-m}\right)} = \frac{(1 + h\mu)\lambda^{1-m}}{1 + h\left(\mu + \mu\gamma N^{*}\lambda^{-m}\right)}\rho(\lambda),$$

where

$$\rho(\lambda) = \lambda^{m} - \frac{1}{1 + h\mu}\lambda^{m-1} + \frac{h\mu\gamma N^{*}}{1 + h\mu},$$

then we only need to show that $\rho(\lambda) > 0$ for $\lambda > 0$. It is not difficult to know that $\rho(\lambda)$ is the characteristic polynomial of the following difference scheme

$$y_{n+1} - y_{n} + \frac{h\mu}{1 + h\mu}y_{n} + \frac{h\mu\gamma N^{*}}{1 + h\mu}y_{n+1-m} = 0.$$

From Theorems 1 and 2, we know that $\rho(\lambda)$ has no positive roots if and only if

$$\frac{h\mu\gamma N^{*}}{1 + h\mu}\frac{m^{m}}{(m-1)^{m-1}} > \left(1 - \frac{h\mu}{1 + h\mu}\right)^{m},$$

which is equivalent to

$$\ln \mu\gamma N^{*}\tau + (m-1)\ln\left(1 + \frac{1 + \mu\tau}{m-1}\right) > 0. \tag{17}$$

(I): If $\mu\gamma N^{*}\tau \geq 1$, then (17) holds true from $m > 1$.

(II): If $\mu\gamma N^{*}\tau < 1$ and $h < {(\tau(1 + \mu\tau + \ln \mu\gamma N^{*}\tau))}\big/{(1 + \mu\tau - \mu\tau \ln \mu\gamma N^{*}\tau)}$, from the fact that " $\ln(1 + x) > x/(1 + x)$ for $x > -1$ and $x \neq 0$ " we get

$$\begin{aligned}\ln \mu\gamma N^{*}\tau + (m-1)\ln\left(1 + \frac{1 + \mu\tau}{m-1}\right) &> \ln \mu\gamma N^{*}\tau + (m-1)\frac{\frac{1 + \mu\tau}{m-1}}{1 + \frac{1 + \mu\tau}{m-1}} \\ &= \ln \mu\gamma N^{*}\tau + \frac{(m-1)(1 + \mu\tau)}{m + \mu\tau} > 0.\end{aligned}$$

Thus Inequality (16) holds for $h < h_{0}$, where h_{0} is defined in (15). From the above discussion, we have

$$B(\lambda) > \lambda - \frac{1}{1 + h\left(\mu + \mu\gamma N^{*}\lambda^{-m}\right)} > 0$$

holds for $h < h_{0}$ and $\lambda > 0$, which implies that (12) has no positive roots. The proof is complete.

In view of (9), Lemmas 2 and 3 and Theorem 1, we have the first main result.

Theorem 4 If condition (5) holds, then (8) is oscillatory for

$$h < \begin{cases} \infty, & when \quad 0 \le \theta \le 1/2, \\ h_0, & when \quad 1/2 < \theta \le 1, \end{cases}$$

where h_0 is defined in (15).

3 Asymptotic Behavior of Non-Oscillatory Solutions

Lemma 4 ([14]) Let $N(t)$ be a positive solution of (1), which does not oscillate about N^*, then $\lim_{t\to\infty} N(t) = N^*$.

From (1) and (6), we know that the non-oscillatory solution of (6) satisfies $\lim_{t\to\infty} x(t) = 0$ if Lemma 4 holds. Next, we will prove that the numerical solution of (1) can preserve this property.

Lemma 5 Let x_n be a non-oscillatory solution of (7), then $\lim_{n\to\infty} x_n = 0$.

Proof: Without loss of generality, we assume $x_n < 0$ for sufficiently large n. Then by (9) we get that $f_1(x_i) < 0$ and $f_2(x_i) < 0$ for sufficiently large i. Moreover, from (7) we have

$$x_{n+1} - x_n + h\theta\mu\gamma N^* f_2(x_{n+1-m}) + h(1-\theta)\mu\gamma N^* f_2(x_{n-m}) = -h\theta\mu f_1(x_{n+1}) - h(1-\theta)\mu f_1(x_n) > 0, \tag{18}$$

hence $x_{n+1} - x_n > 0$, then $\{x_n\}$ is increasing. So there exists an $\eta \le 0$ such that

$$\lim_{n\to\infty} x_n = \eta. \tag{19}$$

Next we will prove $\eta = 0$. If $\eta < 0$, then there exists $N_0 \in \mathbf{N}$ and $\varepsilon > 0$ such that for $n - m > N_0$, $\eta - \varepsilon < x_n < \eta + \varepsilon < 0$. Hence $x_{n-m} < \eta + \varepsilon$ and $x_{n-m+1} < \eta + \varepsilon$. So (18) yields

$$\begin{aligned} 0 &< x_{n+1} - x_n + h\theta\mu\gamma N^* f_2(x_{n+1-m}) + h(1-\theta)\mu\gamma N^* f_2(x_{n-m}) \\ &< x_{n+1} - x_n + h\theta\mu\gamma N^* f_2(\eta+\varepsilon) + h(1-\theta)\mu\gamma N^* f_2(\eta+\varepsilon) \\ &= x_{n+1} - x_n + h\mu\gamma N^* f_2(\eta+\varepsilon), \end{aligned}$$

which implies that $x_{n+1} - x_n > A > 0$, where $A = h\mu\gamma N^*(\exp(-(\eta+\varepsilon)) - 1)$. Thus $x_n \to +\infty$ as $n \to \infty$, which is a contradiction to (19). This ends the proof.

Therefore, the second main result is as follows.

Theorem 5 Let N_n be a positive solution of (8), which does not oscillate about N^*, then $\lim_{n\to\infty} N_n = N^*$.

4 Numerical Examples

In this section, we will give four examples to illustrate the main results.

Example 1 Consider the equation

$$N'(t) = -N(t) + 2e^{-0.5N(t-1.8)}, \quad t \geq 0 \tag{20}$$

with initial value $N(t) = 2.5$ for $-1.8 \leq t \leq 0$. In (20), $\mu\gamma N^{*}\tau e^{\mu\tau} \approx 6.1525 > 1/e$, so condition (5) holds true. That is, the analytic solutions of (20) are oscillatory. Let $m = 18$ and $\theta = 0.3 \in [0, 1/2]$, we draw the figures of the analytic solutions and the numerical solutions of (20) in Fig. 1. From this figure we can see that the numerical solutions of (20) oscillate about $N^{*} \approx 1.13$, which coincides with Theorem 4.

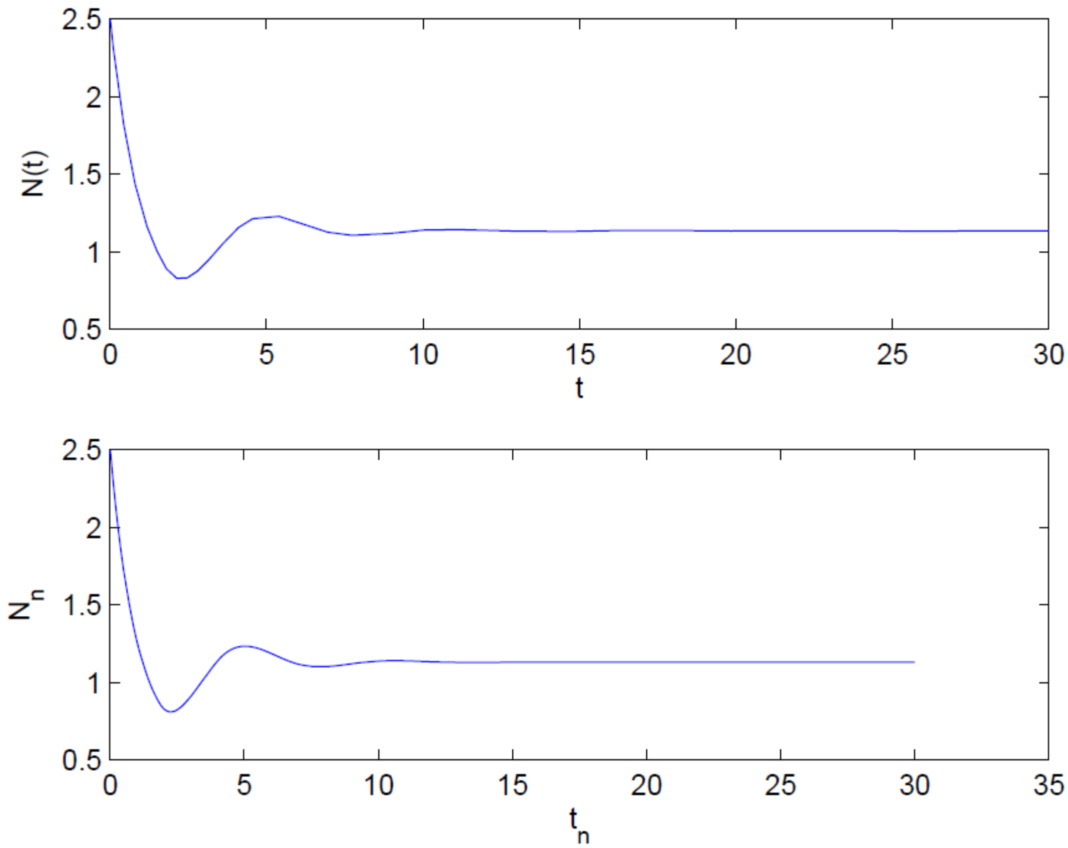

Fig. 1 The analytic solution and the numerical solution of (20) with $\theta = 0.3$ and $m = 18$.

Example 2 For the equation

$$N'(t) = -N(t) + 2e^{-0.5N(t-2.5)}, \quad t \geq 0 \tag{21}$$

with initial value $N(t) = 0.2$ for $-2.5 \le t \le 0$, it is easy to verify that condition (5) holds. That is, the analytic solutions of (21) are oscillatory. In Fig. 2, we draw the figures of the analytic solutions and the numerical solutions of (21), respectively. We set $m = 50, \theta = 0.8 \in [1/2, 1]$. So $\mu\gamma N^*\tau \approx 1.4125 \ge 1$ and $h = \tau / m = 0.05 < h_0 = \infty$. From this figure, we can see that the numerical solutions of (21) oscillate about $N^* \approx 1.13$, which are in agreement with Theorem 4.

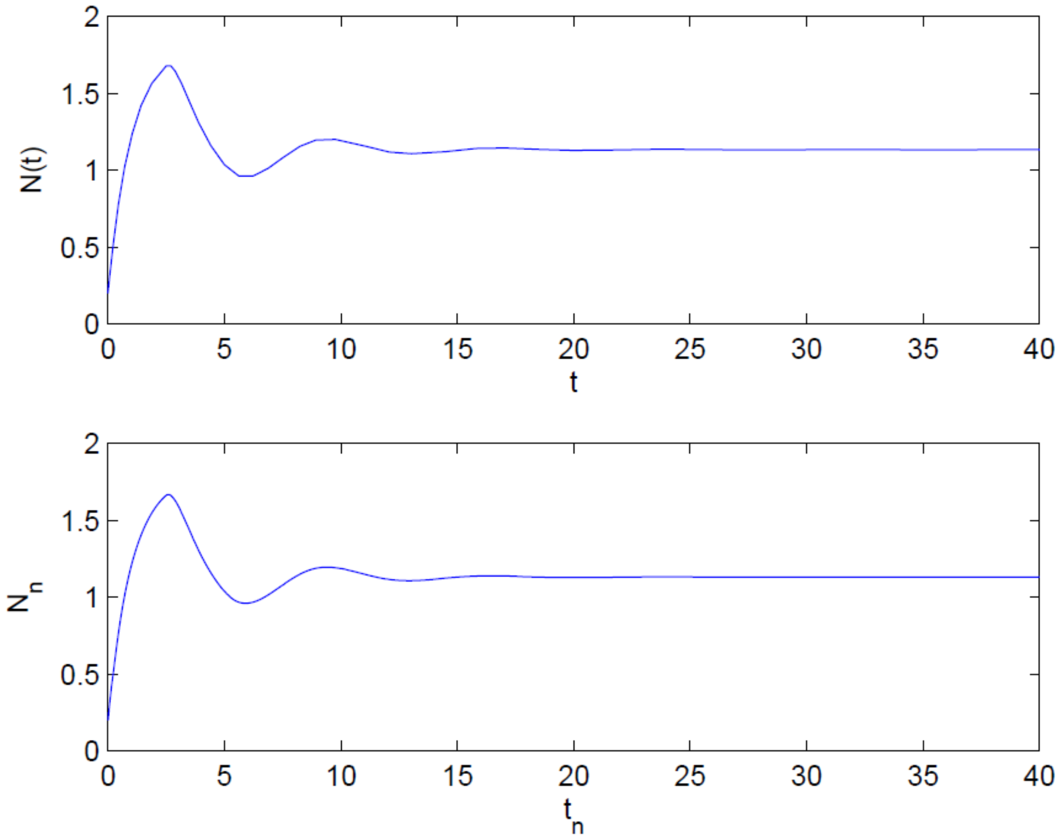

Fig. 2 The analytic solution and the numerical solution of (21) with $\theta = 0.8$ and $m = 50$.

Example 3 We consider the equation

$$N'(t) = -N(t) + 2e^{-0.5N(t-0.7)}, \quad t \ge 0 \tag{22}$$

with initial value $N(t) = 1.3$ for $-0.7 \le t \le 0$. In (22), it is not difficult to see that condition (5) is fulfilled. That is, the analytic solutions of (22) are oscillatory. In Fig. 3, we draw the figures of the analytic solutions and the numerical solutions of (22), respectively. Let $m = 14$, $\theta = 0.6 \in [1/2, 1]$ and $\mu\gamma N^*\tau \approx 0.3955 < 1$. So $h = \tau / m = 0.05 < h_0 \approx 0.2301$. We can see from this figure that the numerical solutions of (22) oscillate about $N^* \approx 1.13$, which are consistent with Theorem 4.

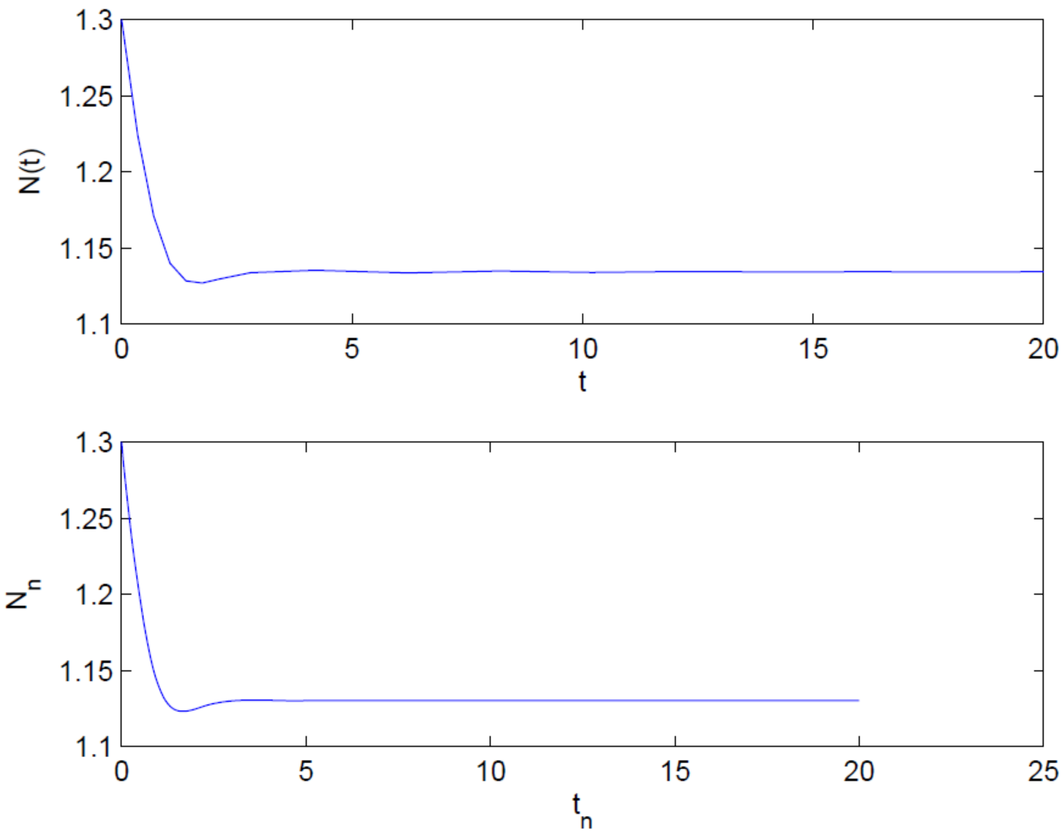

Fig. 3 The analytic solution and the numerical solution of (22) with $\theta = 0.6$ and $m = 14$.

Example 4 For the equation

$$N'(t) = -N(t) + 0.3e^{-1.6N(t-0.5)}, \quad t \geq 0 \tag{23}$$

with initial value $N(t) = 5$ for $-0.5 \leq t \leq 0$, it is easy to see that $\mu\gamma N^*\tau e^{\mu\tau} \approx 0.2809 < 1/e$, so condition (5) is not satisfied. That is, the analytic solutions of (23) are non-oscillatory. In Fig. 4, we draw the figures of the analytic solutions and the numerical solutions of (23), respectively. In this figure, we can see that $N(t) \to N^* \approx 0.213$ as $t \to \infty$ and the numerical solutions of (23) also satisfy $N_n \to N^* \approx 0.213$ as $n \to \infty$. That is, the numerical method preserves the asymptotic property of non-oscillatory solutions of (23), which coincides with Theorem 5.

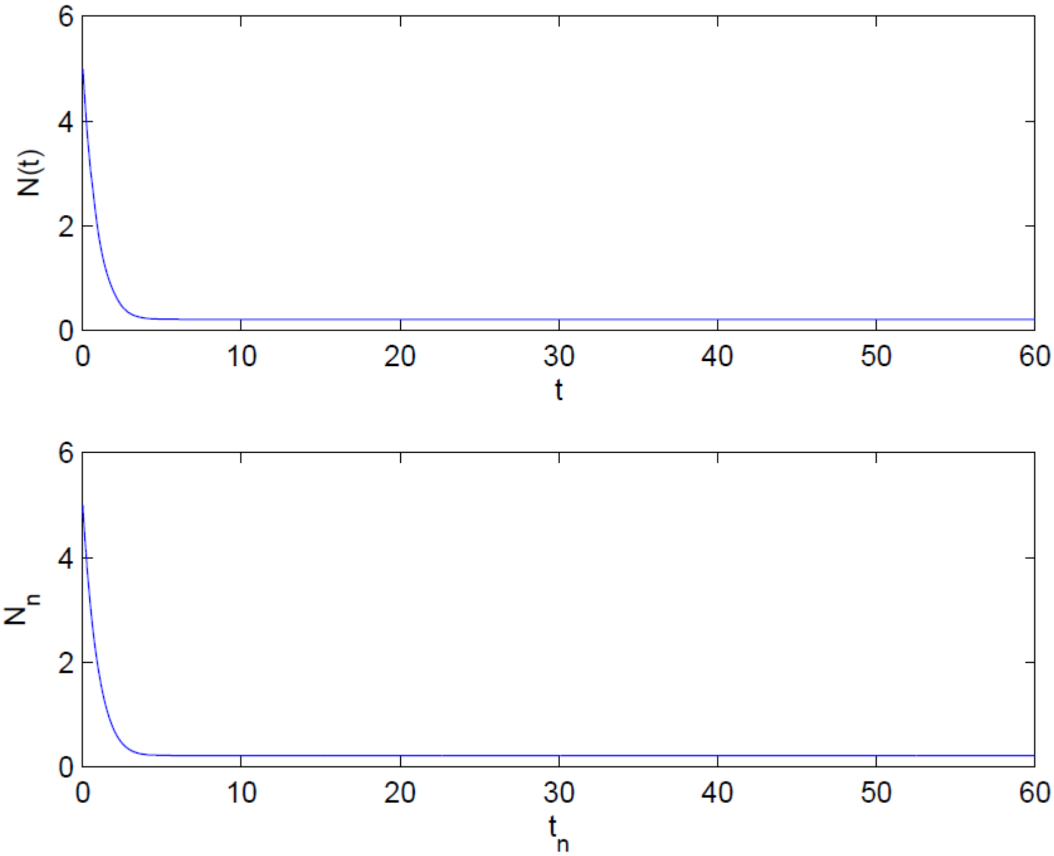

Fig. 4 The analytic solution and the numerical solution of (23) with $\theta = 0.2$ and $m = 20$.

Moreover, we can see from these figures that the θ-methods preserve the oscillation of (20)-(22) and the non-oscillation of (23), respectively.

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