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A Study On Generalized (*r, s, t, u, v, y*)-Numbers

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Abstract. In this paper, we introduce the generalized (r,s,t,u,v,y) sequence and we deal with, in detail, three special cases which we call them (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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Keywords. Hexanacci numbers, (r,s,t,u,v,y) numbers, Lucas (r,s,t,u,v,y) numbers, modified (r,s,t,u,v,y) numbers.

1. Introduction

The generalized (r,s,t,u,v,y) sequence (or the generalized Hexanacci sequence or 6-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;r,s,t,u,v,y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth-order recurrence relation

$$\begin{aligned} W_n &= rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}+yW_{n-6}, \qquad (1.1)\\ W_0 &= c_0, W_1=c_1, W_2=c_2,\ W_3=c_3, W_4=c_4, W_5=c_5,\ n\geq 6 \end{aligned}$$

where W_0,W_1,W_2,W_3,W_4,W_5 are arbitrary real or complex numbers and r,s,t,u,v,y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=-\frac{v}{y}W_{-n+1}-\frac{u}{y}W_{-n+2}-\frac{t}{y}W_{-n+3}-\frac{s}{y}W_{-n+4}-\frac{r}{y}W_{-n+5}+\frac{1}{y}W_{-n+6}$$

for $n=1,2,3,\ldots$ when $r_6\neq 0$. Therefore, recurrence (1.1) holds for all integers n. Hexanacci sequence has been studied by many authors, see for example [3], [4].

In literature, for example, the following names and notations (see Table 1) are used for the special case of r,s,t,u,y and initial values.

Table 1. A few special case of generalized Hexanacci sequences.

Sequences (Numbers)	Notation	OEIS [5]	Ref
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592	[7]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584	[7]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$		[8]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$		[8]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$		[8]
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$		[9]
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$		[9]
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$		[9]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

As $\{W_n\}$ is a sixth-order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - rx^5 - sx^4 - tx^3 - ux^2 - vx - y = 0 \tag{1.2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

Note that we have the following identities:

$\alpha + \beta + \gamma + \delta + \lambda + \mu = r,$

$\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta = -s,$

$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta = t,$

$\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta = -u$

$\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta = v,$

$\alpha\beta\lambda\gamma\mu\delta = -y.$

Throughout the paper, we use the following notations interchangeable.

$$r = r_1, s = r_2, t = r_3, u = r_4, v = r_5, y = r_6$$

and

$$\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3, \delta = \alpha_4, \lambda = \alpha_5, \mu = \alpha_6.$$

So (1.1) and (1.2) can be written as follows:

$$W_n = r_1W_{n-1} + r_2W_{n-2} + r_3W_{n-3} + r_4W_{n-4} + r_5W_{n-5} + r_6W_{n-6}$$

and

$$x^6 - r_1x^5 - r_2x^4 - r_3x^3 - r_4x^2 - r_5x - r_6 = 0$$

respectively.

Generalized Hexanacci numbers can be expressed, for all integers n, using Binet's formula.

THEOREM 1. *(Binet's formula of generalized (r,s,t,u,v,y) numbers (generalized Hexanacci numbers))*

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &\quad +\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &\quad +\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{p_6\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)} \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} p_1 &= W_5-(\beta+\gamma+\delta+\lambda+\mu)W_4+(\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ &\quad -(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\ &\quad +(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\beta\lambda\gamma\mu\delta W_0, \\ p_2 &= W_5-(\alpha+\gamma+\delta+\lambda+\mu)W_4+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ &\quad -(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\ &\quad +(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\alpha\lambda\gamma\mu\delta W_0, \\ p_3 &= W_5-(\alpha+\beta+\delta+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)W_3 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)W_2 \\ &\quad +(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)W_1-\alpha\beta\lambda\mu\delta W_0, \\ p_4 &= W_5-(\alpha+\beta+\gamma+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)W_3 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)W_2 \\ &\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)W_1-\alpha\beta\lambda\gamma\mu W_0, \\ p_5 &= W_5-(\alpha+\beta+\gamma+\delta+\mu)W_4+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ &\quad -(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)W_2 \\ &\quad +(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)W_1-\alpha\beta\gamma\mu\delta W_0, \\ p_6 &= W_5-(\alpha+\beta+\gamma+\delta+\lambda)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_3 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_2 \\ &\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)W_1-\alpha\beta\lambda\gamma\delta W_0. \end{aligned}$$

Usually, it is customary to choose $r_1, r_2, r_3, r_4, r_5, r_6$ so that the Equ. (1.2) has at least one real (say α_1) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, (see [1], this result of Howard and Saidak [1] is even true in the case of higher-order recurrence relations).

(1.3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n + A_6\mu^n$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)},\\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)},\\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)},\\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)},\\
A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)},\\
A_6 &= \frac{p_6}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}.
\end{aligned}$$

Note that (1.3) can also be written in the following form:

$$W_n = \sum_{k=1}^{6} \frac{p_k \alpha_k^n}{\prod\limits_{\substack{j=1\\ k\neq j}}^{6} (\alpha_k - \alpha_j)}.$$

We have the following formula: for $n = 1, 2, 3, ...$ we have

$$W_{-n} = \frac{\alpha^{-n}X_1p_1 + \beta^{-n}X_2p_2 + \gamma^{-n}X_3p_3 + \delta^{-n}X_4p_4 + \lambda^{-n}X_5p_5 + \mu^{-n}X_6p_6}{\alpha^{n}X_1p_1 + \beta^{n}X_2p_2 + \gamma^{n}X_3p_3 + \delta^{n}X_4p_4 + \lambda^{n}X_5p_5 + \mu^{n}X_6p_6} W_n$$

where

$$\begin{aligned}
X_1 &= (\mu-\delta)(\gamma-\delta)(\gamma-\mu)(\lambda-\delta)(\lambda-\mu)(\lambda-\gamma)(\beta-\delta)(\beta-\mu)(\beta-\gamma)(\beta-\lambda),\\
X_2 &= (\delta-\mu)(\gamma-\delta)(\gamma-\mu)(\lambda-\delta)(\lambda-\mu)(\lambda-\gamma)(\alpha-\delta)(\alpha-\mu)(\alpha-\gamma)(\alpha-\lambda),\\
X_3 &= (\delta-\mu)(\lambda-\delta)(\lambda-\mu)(\beta-\delta)(\beta-\mu)(\beta-\lambda)(\alpha-\delta)(\alpha-\mu)(\alpha-\lambda)(\alpha-\beta),\\
X_4 &= (\mu-\gamma)(\lambda-\mu)(\lambda-\gamma)(\beta-\mu)(\beta-\gamma)(\beta-\lambda)(\alpha-\mu)(\alpha-\gamma)(\alpha-\lambda)(\alpha-\beta),\\
X_5 &= (\mu-\delta)(\gamma-\delta)(\gamma-\mu)(\beta-\delta)(\beta-\mu)(\beta-\gamma)(\alpha-\delta)(\alpha-\mu)(\alpha-\gamma)(\alpha-\beta),\\
X_6 &= (\gamma-\delta)(\lambda-\delta)(\lambda-\gamma)(\beta-\delta)(\beta-\gamma)(\beta-\lambda)(\alpha-\delta)(\alpha-\gamma)(\alpha-\lambda)(\alpha-\beta).
\end{aligned}$$

We can also give Binet's formula of the generalized (r,s,t,u,v,y) numbers (the generalized Hexanacci numbers) for the negative subscripts as follows: for $n = 1, 2, 3, ...$ we have

$$\begin{aligned}
W_{-n} &= \frac{\alpha^5 - r\alpha^4 - s\alpha^3 - t\alpha^2 - u\alpha - v}{y(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} p_1\alpha^{1-n} + \frac{\beta^5 - r\beta^4 - s\beta^3 - t\beta^2 - u\beta - v}{y(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} p_2\beta^{1-n}\\
&\quad + \frac{\gamma^5 - r\gamma^4 - s\gamma^3 - t\gamma^2 - u\gamma - v}{y(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} p_3\gamma^{1-n} + \frac{\delta^5 - r\delta^4 - s\delta^3 - t\delta^2 - u\delta - v}{y(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} p_4\delta^{1-n}\\
&\quad + \frac{\lambda^5 - r\lambda^4 - s\lambda^3 - t\lambda^2 - u\lambda - v}{y(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} p_5\lambda^{1-n} + \frac{\mu^5 - r\mu^4 - s\mu^3 - t\mu^2 - u\mu - v}{y(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)} p_6\mu^{1-n}.
\end{aligned}$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n. The following lemma is a special case of a well known formula of generating functions of the generalized m-step Fibonacci numbers which can be found in the literature (see for example [10]). For completeness, we include the proof.

LEMMA 2. *Suppose that* $f_{W_n}(x)=\sum\limits_{n=0}^{\infty}W_nx^n$ *is the ordinary generating function of the generalized* (r,s,t,u,v,y) *sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty}W_nx^n$ *is given by*

$$\sum_{n=0}^{\infty}W_nx^n=\frac{\Lambda}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6}=\frac{\Lambda}{1-r_1x-r_2x^2-r_3x^3-r_4x^4-r_5x^5-r_6x^6} \tag{1.4}$$

where

$$\begin{aligned}\Lambda &= W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3\\ &\quad +(W_4-rW_3-sW_2-tW_1-uW_0)x^4+(W_5-rW_4-sW_3-tW_2-uW_1-vW_0)x^5\end{aligned}$$

or (in short formula)

$$\begin{aligned}\Lambda &= W_0+(W_1-r_1W_0)x+(W_2-r_1W_1-r_2W_0)x^2+(W_3-r_1W_2-r_2W_1-r_3W_0)x^3\\ &\quad +(W_4-r_1W_3-r_2W_2-r_3W_1-r_4W_0)x^4+(W_5-r_1W_4-r_2W_3-r_3W_2-r_4W_1-r_5W_0)x^5\\ &= W_0+\sum_{i=1}^{6-1}x^i\left(W_i-\sum_{j=1}^{i}r_jW_{i-j}\right).\end{aligned}$$

Proof. Using the definition of generalized 6-primes numbers, and substracting $r_1x\sum_{n=0}^{\infty}W_nx^n$, $r_2x^2\sum_{n=0}^{\infty}W_nx^n$, $r_3x^3\sum_{n=0}^{\infty}W_nx^n$, $r_4x^4\sum_{n=0}^{\infty}W_nx^n$, $r_5x^5\sum_{n=0}^{\infty}W_nx^n$ and $r_6x^6\sum_{n=0}^{\infty}W_nx^n$from $\sum_{n=0}^{\infty}W_nx^n$ we obtain

$$\begin{aligned}&(1-r_1x-r_2x^2-r_3x^3-r_4x^4-r_5x^5-r_6x^6)\sum_{n=0}^{\infty}W_nx^n\\
&= \sum_{n=0}^{\infty}W_nx^n-r_1x\sum_{n=0}^{\infty}W_nx^n-r_2x^2\sum_{n=0}^{\infty}W_nx^n-r_3x^3\sum_{n=0}^{\infty}W_nx^n-r_4x^4\sum_{n=0}^{\infty}W_nx^n\\
&\quad -r_5x^5\sum_{n=0}^{\infty}W_nx^n-r_6x^6\sum_{n=0}^{\infty}W_nx^n\\
&= \sum_{n=0}^{\infty}W_nx^n-r_1\sum_{n=0}^{\infty}W_nx^{n+1}-r_2\sum_{n=0}^{\infty}W_nx^{n+2}-r_3\sum_{n=0}^{\infty}W_nx^{n+3}-r_4\sum_{n=0}^{\infty}W_nx^{n+4}\\
&\quad -r_5\sum_{n=0}^{\infty}W_nx^{n+5}-r_6\sum_{n=0}^{\infty}W_nx^{n+6}\\
&= \sum_{n=0}^{\infty}W_nx^n-r_1\sum_{n=1}^{\infty}W_{n-1}x^n-r_2\sum_{n=2}^{\infty}W_{n-2}x^n-r_3\sum_{n=3}^{\infty}W_{n-3}x^n-r_4\sum_{n=4}^{\infty}W_{n-4}x^n\\
&\quad -r_5\sum_{n=5}^{\infty}W_{n-5}x^n-r_6\sum_{n=6}^{\infty}W_{n-6}x^n\\
&= (W_0+W_1x+W_2x^2+W_3x^3+W_4x^4+W_5x^5)-r_1(W_0x+W_1x^2+W_2x^3+W_3x^4+W_4x^5)\\
&\quad -r_2(W_0x^2+W_1x^3+W_2x^4+W_3x^5)-r_3(W_0x^3+W_1x^4+W_2x^5)-r_4(W_0x^4+W_1x^5)-r_5W_0x^5\\
&\quad +\sum_{n=6}^{\infty}(W_n-r_1W_{n-1}-r_2W_{n-2}-r_3W_{n-3}-r_4W_{n-4}-r_5W_{n-5}-r_6W_{n-6})x^n\\
&= W_0+(W_1-r_1W_0)x+(W_2-r_1W_1-r_2W_0)x^2+(W_3-r_1W_2-r_2W_1-r_3W_0)x^3\\
&\quad +(W_4-r_1W_3-r_2W_2-r_3W_1-r_4W_0)x^4+(W_5-r_1W_4-r_2W_3-r_3W_2-r_4W_1-r_5W_0)x^5\\
&= W_0+\sum_{i=1}^{6-1}x^i\left(W_i-\sum_{j=1}^{i}r_jW_{i-j}\right).\end{aligned}$$

Rearranging above equation, we obtain (1.4). □

We next find Binet's formula of generalized (r, s, t, u, v, y) numbers $\{W_n\}$ by the use of generating function for W_n.

THEOREM 3. *(Binet's formula of generalized (r, s, t, u, v, y) numbers)*

$$W_n = \sum_{k=1}^{6} \frac{q_k \alpha_k^n}{\prod\limits_{\substack{j=1 \\ k \neq j}}^{6} (\alpha_k - \alpha_j)} \tag{1.5}$$

where

$$q_l = W_0 \alpha_l^{6-1} + \sum_{i=1}^{6-1} \alpha_l^{6-1-i} \left[W_i - \sum_{j=1}^{i} r_j W_{i-j} \right], \quad 1 \leq l \leq m = 6.$$

Proof. Let

$$h(x) = 1 - r_1 x - r_2 x^2 - r_3 x^3 - r_4 x^4 - r_5 x^5 - r_6 x^6.$$

Then for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 we write

$$h(x) = (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x)$$

i.e.,

$$1 - r_1 x - r_2 x^2 - r_3 x^3 - r_4 x^4 - r_5 x^5 - r_6 x^6 = (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x). \tag{1.6}$$

Hence $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \frac{1}{\alpha_4}, \frac{1}{\alpha_5}$ and $\frac{1}{\alpha_6}$ are the roots of $h(x)$. This gives $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r_1}{x} - \frac{r_2}{x^2} - \frac{r_3}{x^3} - \frac{r_4}{x^4} - \frac{r_5}{x^5} - \frac{r_6}{x^6} = 0.$$

This implies $x^6 - r_1 x^5 - r_2 x^4 - r_3 x^3 - r_4 x^2 - r_5 x - r_6 = 0$. Now, by (1.4) and (1.6), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + \sum_{i=1}^{6-1} x^i \left[W_i - \sum_{j=1}^{i} r_j W_{i-j} \right]}{(1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x)}.$$

Then we write

$$\begin{aligned} \frac{W_0 + \sum_{i=1}^{6-1} x^i \left[W_i - \sum_{j=1}^{i} r_j W_{i-j} \right]}{(1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x)} &= \frac{B_1}{(1 - \alpha_1 x)} + \frac{B_2}{(1 - \alpha_2 x)} + \frac{B_3}{(1 - \alpha_3 x)} \\ &\quad + \frac{B_4}{(1 - \alpha_4 x)} + \frac{B_5}{(1 - \alpha_5 x)} + \frac{B_6}{(1 - \alpha_6 x)}. \end{aligned} \tag{1.7}$$

So

$$\begin{aligned} & W_0 + \sum_{i=1}^{6-1} x^i \left[W_i - \sum_{j=1}^{i} r_j W_{i-j} \right] \\ = \;& B_1 (1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x) + B_2 (1 - \alpha_1 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x) \\ & + B_3 (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_4 x)(1 - \alpha_5 x)(1 - \alpha_6 x) + B_4 (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_5 x)(1 - \alpha_6 x) \\ & + B_5 (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_6 x) + B_6 (1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x)(1 - \alpha_5 x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha_1}$, we get

$$W_0 + \sum_{i=1}^{6-1} \left(\frac{1}{\alpha_1} \right)^i \left[W_i - \sum_{j=1}^{i} r_j W_{i-j} \right] = B_1 (1 - \frac{\alpha_2}{\alpha_1})(1 - \frac{\alpha_3}{\alpha_1})(1 - \frac{\alpha_4}{\alpha_1})(1 - \frac{\alpha_5}{\alpha_1})(1 - \frac{\alpha_6}{\alpha_1}).$$

This gives

$$\begin{aligned} B_1 &= \frac{\alpha_1^5(W_0+\sum_{i=1}^{6-1}\left(\frac{1}{\alpha_1}\right)^i\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right])}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)} \\ &= \frac{W_0\alpha_1^{6-1}+\sum_{i=1}^{6-1}\alpha_1^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{W_0\alpha_2^{6-1}+\sum_{i=1}^{6-1}\alpha_2^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_2-\alpha_5)(\alpha_2-\alpha_6)}, \\ B_3 &= \frac{W_0\alpha_3^{6-1}+\sum_{i=1}^{6-1}\alpha_3^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)(\alpha_3-\alpha_5)(\alpha_3-\alpha_6)}, \\ B_4 &= \frac{W_0\alpha_4^{6-1}+\sum_{i=1}^{6-1}\alpha_4^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_4-\alpha_5)(\alpha_4-\alpha_6)}, \\ B_5 &= \frac{W_0\alpha_5^{6-1}+\sum_{i=1}^{6-1}\alpha_5^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_5-\alpha_1)(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)(\alpha_5-\alpha_6)}, \\ B_6 &= \frac{W_0\alpha_6^{6-1}+\sum_{i=1}^{6-1}\alpha_6^{6-1-i}\left[W_i-\sum_{j=1}^{i}r_jW_{i-j}\right]}{(\alpha_6-\alpha_1)(\alpha_6-\alpha_2)(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5)}. \end{aligned}$$

Thus (1.7) can be written as

$$\sum_{n=0}^{\infty}W_nx^n = B_1(1-\alpha_1x)^{-1}+B_2(1-\alpha_2x)^{-1}+B_3(1-\alpha_3x)^{-1}+B_4(1-\alpha_4x)^{-1}+B_5(1-\alpha_5x)^{-1}+B_6(1-\alpha_6x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty}W_nx^n &= B_1\sum_{n=0}^{\infty}\alpha_1^nx^n+B_2\sum_{n=0}^{\infty}\alpha_2^nx^n+B_3\sum_{n=0}^{\infty}\alpha_3^nx^n+B_4\sum_{n=0}^{\infty}\alpha_4^nx^n+B_5\sum_{n=0}^{\infty}\alpha_5^nx^n+B_6\sum_{n=0}^{\infty}\alpha_6^nx^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha_1^n+B_2\alpha_2^n+B_3\alpha_3^n+B_4\alpha_4^n+B_5\alpha_5^n+B_6\alpha_6^n)x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha_1^n+B_2\alpha_2^n+B_3\alpha_3^n+B_4\alpha_4^n+B_5\alpha_5^n+B_6\alpha_6^n$$

and then we get (1.5). □

In this paper, we define and investigate, in detail, three special cases of the generalized (r,s,t,u,v,y) sequence $\{W_n\}$ which we call them (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) sequences. (r,s,t,u,v,y) sequence$\{G_n\}_{n\geq 0}$, Lucas (r,s,t,u,v,y) sequence $\{H_n\}_{n\geq 0}$ and modified (r,s,t,u,v,y) sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the sixth-order recurrence relations

$$
\begin{aligned}
G_{n+6} &= rG_{n+5}+sG_{n+4}+tG_{n+3}+uG_{n+2}+vG_{n+1}+yG_n,\\
G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, G_4=r^3+2sr+t, G_5=r^4+s^2+3r^2s+2rt+u,\\
H_{n+6} &= rH_{n+5}+sH_{n+4}+tH_{n+3}+uH_{n+2}+vH_{n+1}+yH_n,\\
H_0 &= 6, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t, H_4=r^4+4r^2s+4tr+2s^2+4u,\\
H_5 &= r^5+5r^3s+5tr^2+5rs^2+5ur+5ts+5v,\\
E_{n+6} &= rE_{n+5}+sE_{n+4}+tE_{n+3}+uE_{n+2}+vE_{n+1}+yE_n,\\
E_0 &= 1, E_1=r-1, E_2=-r+s+r^2, E_3=r^3-r^2+2sr-s+t,\\
E_4 &= r^4-r^3+3r^2s-2rs+2tr+s^2-t+u,\\
E_5 &= r^5-r^4-s^2+3rs^2-3r^2s+4r^3s+3r^2t-2rt+2ru+2st+v-u.
\end{aligned}
$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{v}{y}G_{-(n-1)}-\frac{u}{y}G_{-(n-2)}-\frac{t}{y}G_{-(n-3)}-\frac{s}{y}G_{-(n-4)}-\frac{r}{y}G_{-(n-5)}+\frac{1}{y}G_{-(n-6)}, \tag{1.8}$$

$$H_{-n} = -\frac{v}{y}H_{-(n-1)}-\frac{u}{y}H_{-(n-2)}-\frac{t}{y}H_{-(n-3)}-\frac{s}{y}H_{-(n-4)}-\frac{r}{y}H_{-(n-5)}+\frac{1}{y}H_{-(n-6)}, \tag{1.9}$$

$$E_{-n} = -\frac{v}{y}E_{-(n-1)}-\frac{u}{y}E_{-(n-2)}-\frac{t}{y}E_{-(n-3)}-\frac{s}{y}E_{-(n-4)}-\frac{r}{y}E_{-(n-5)}+\frac{1}{y}E_{-(n-6)}, \tag{1.10}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.8), (1.9) and (1.10) hold for all integers n.

Next, we present the first few values of the (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) numbers with positive and negative subscripts:

Table 2. The first few values of the special sixth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5
G_n	0	1	r	r^2+s	$r^3+2sr+t$	G_5
G_{-n}		0	0	0	0	G_{-5}
H_n	6	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2+4u$	H_5
H_{-n}		$-\frac{v}{y}$	$\frac{1}{y^2}(-2uy+v^2)$	$-\frac{1}{y^3}(3ty^2+v^3-3uvy)$	$\frac{1}{y^4}(-4sy^3+2u^2y^2+v^4+4tvy^2-4uv^2y)$	H_{-5}
E_n	1	$r-1$	$-r+s+r^2$	$r^3-r^2+2sr-s+t$	$r^4-r^3+3r^2s-2rs+2tr+s^2-t+u$	E_5
E_{-n}		0	0	0	0	E_{-5}

where

$$
\begin{aligned}
G_5 &= r^4+s^2+3r^2s+2rt+u,\\
G_{-5} &= \frac{1}{y},\\
H_5 &= r^5+5r^3s+5tr^2+5rs^2+5ur+5ts+5v,\\
H_{-5} &= -\frac{1}{y^5}(5ry^4+v^5-5svy^3-5tuy^3-5uv^3y+5tv^2y^2+5u^2vy^2),\\
E_5 &= r^5-r^4-s^2+3rs^2-3r^2s+4r^3s+3r^2t-2rt+2ru+2st+v-u,\\
E_{-5} &= -\frac{1}{y}.
\end{aligned}
$$

Some special cases of (r,s,t,u,v,y) sequence $\{G_n(0,1,r,r^2+s,r^3+2sr+t,r^4+s^2+3r^2s+2rt+u;r,s,t,u,v,y)\}$ and Lucas (r,s,t,u,v,y) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t,r^4+4r^2s+4tr+2s^2+4u,r^5+5r^3s+5tr^2+5rs^2+5ur+5ts+5v$ $;r,s,t,u,v,y)\}$ are as follows:

(1) $G_n(0,1,1,2,4,8;1,1,1,1,1,1)=H_n$, Hexanacci sequence,
(2) $H_n(6,1,3,7,15,31;1,1,1,1,1,1)=E_n$, Hexanacci-Lucas sequence,
(3) $G_n(0,1,2,5,13,34;2,1,1,1,1,1)=P_n$, sixth-order Pell sequence,
(4) $H_n(6,2,6,17,46,122;2,1,1,1,1,1)=Q_n$, sixth-order Pell-Lucas sequence.

For all integers n, (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) numbers (using initial conditions in (1.3) or (1.5)) can be expressed, respectively, using Binet's formulas as

$$\begin{aligned}
G_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)}\\
&\quad+\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)}\\
&\quad+\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},\\
H_n &= \alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n+\mu^n,\\
E_n &= \frac{(\alpha-1)\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(\beta-1)\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)}\\
&\quad+\frac{(\gamma-1)\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(\delta-1)\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)}\\
&\quad+\frac{(\lambda-1)\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(\mu-1)\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}$$

or (in short formulas)

$$\begin{aligned}
G_n &= \sum_{k=1}^{6}\frac{\alpha_k^{n+4}}{\prod\limits_{\substack{j=1\\k\neq j}}^{6}(\alpha_k-\alpha_j)},\\
H_n &= \sum_{k=1}^{6}\alpha_k^n=\alpha_1^n+\alpha_2^n+\alpha_3^n+\alpha_4^n+\alpha_5^n+\alpha_6^n,\\
E_n &= \sum_{k=1}^{6}\frac{(\alpha_k-1)\alpha_k^{n+4}}{\prod\limits_{\substack{j=1\\k\neq j}}^{6}(\alpha_k-\alpha_j)}.
\end{aligned}$$

Note that for all n we have

$$E_n=G_{n+1}-G_n.$$

Lemma 2 gives the following results as particular examples (generating functions of (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) numbers).

COROLLARY 4. *Generating functions of* (r,s,t,u,v,y)*, Lucas* (r,s,t,u,v,y) *and modified* (r,s,t,u,v,y) *numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty}G_nx^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6},\\ \sum_{n=0}^{\infty}H_nx^n &= \frac{6-5rx-4sx^2-3tx^3-2ux^4-vx^5}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6},\\ \sum_{n=0}^{\infty}E_nx^n &= \frac{1-x}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6},\end{aligned}$$

respectively.

Proof. In Lemma 2, take $W_n=G_n$ with $G_0=0,G_1=1,G_2=r,G_3=r^2+s,G_4=r^3+2sr+t,G_5=r^4+s^2+3r^2s+2rt+u$, $W_n=H_n$ with $H_0=4,H_1=r,H_2=2s+r^2,H_3=r^3+3sr+3t,H_4=r^4+4r^2s+4tr+2s^2+4u,H_5=r^5+5r^3s+5tr^2+5rs^2+5ur+5ts+5v$, and $W_n=E_n$ with $E_0=1,E_1=r-1,E_2=-r+s+r^2,E_3=r^3-r^2+2sr-s+t,E_4=r^4-r^3+3r^2s-2rs+2tr+s^2-t+u,E_5=r^5-r^4-s^2+3rs^2-3r^2s+4r^3s+3r^2t-2rt+2ru+2st+v-u$, respectively. □

2. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1}\end{vmatrix}=(-1)^n.$$

The following theorem gives generalization of this result to the generalized (r,s,t,u,v,y) sequence $\{W_n\}_{n\geq 0}$.

THEOREM 5 (Simson Formula of Generalized (r,s,t,u,v,y)-Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+5} & W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} & W_{n-5}\end{vmatrix} = (-1)^n y^n \begin{vmatrix} W_5 & W_4 & W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_3 & W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} & W_{-5}\end{vmatrix}.$$

Proof. It is given in Soykan [6]. □

The previous theorem gives the following results as particular examples.

COROLLARY 6. *For all integers n, Simson formula of (r, s, t, u, v, y), Lucas (r, s, t, u, v, y) and modified (r, s, t, u, v, y) numbers are given as*

$$\begin{vmatrix} G_{n+5} & G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} & G_{n-5} \end{vmatrix} = (-1)^n y^{n-1},$$

$$\begin{vmatrix} H_{n+5} & H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} & H_{n-5} \end{vmatrix} = (-1)^n y^{n-5} g(r, s, t, u, v, y),$$

$$\begin{vmatrix} E_{n+5} & E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} & E_{n-5} \end{vmatrix} = (-1)^{n+1} y^{n-1}(r + s + t + u + v + y - 1),$$

where

$$\begin{aligned}
g(r, s, t, u, v, y) = {} & -3125r^6y^4 + 2500r^5svy^3 + 3750r^5tuy^3 - 2000r^5tv^2y^2 - 2250r^5u^2vy^2 + 1600r^5uv^3y - 256r^5v^5 - \\
& 2000r^4s^2uy^3 + 50r^4s^2v^2y^2 - 2250r^4st^2y^3 + 2050r^4stuvy^2 - 160r^4stv^3y + 900r^4su^3y^2 - 1020r^4su^2v^2y + 192r^4suv^4 - \\
& 22\,500r^4sy^4 + 900r^4t^3vy^2 - 825r^4t^2u^2y^2 - 560r^4t^2uv^2y + 128r^4t^2v^4 + 630r^4tu^3vy - 144r^4tu^2v^3 + 2250r^4tvy^3 - 108r^4u^5y + \\
& 27r^4u^4v^2 + 1500r^4u^2y^3 - 1700r^4uv^2y^2 + 320r^4v^4y + 1600r^3s^3ty^3 - 160r^3s^3uvy^2 + 36r^3s^3v^3y - 1020r^3s^2t^2vy^2 - \\
& 560r^3s^2tu^2y^2 + 746r^3s^2tuv^2y - 144r^3s^2tv^4 - 24r^3s^2u^3vy + 6r^3s^2u^2v^3 + 15\,600r^3s^2vy^3 + 630r^3st^3uy^2 - 24r^3st^3v^2y - \\
& 356r^3st^2u^2vy + 80r^3st^2uv^3 + 72r^3stu^4y - 18r^3stu^3v^2 + 19\,800r^3stuy^3 - 12\,330r^3stv^2y^2 - 13\,040r^3su^2vy^2 + 9768r^3suv^3y - \\
& 1600r^3sv^5 - 108r^3t^5y^2 + 72r^3t^4uvy - 16r^3t^4v^3 - 16r^3t^3u^3y + 4r^3t^3u^2v^2 - 1350r^3t^3y^3 + 1980r^3t^2uvy^2 - 208r^3t^2v^3y - \\
& 120r^3tu^3y^2 - 682r^3tu^2v^2y + 160r^3tuv^4 - 27\,000r^3ty^4 + 144r^3u^4vy - 36r^3u^3v^3 + 1800r^3uvy^3 - 410r^3v^3y^2 - 256r^2s^5y^3 + \\
& 192r^2s^4tvy^2 + 128r^2s^4u^2y^2 - 144r^2s^4uv^2y + 27r^2s^4v^4 - 144r^2s^3t^2uy^2 + 6r^2s^3t^2v^2y + 80r^2s^3tu^2vy - 18r^2s^3tuv^3 - \\
& 16r^2s^3u^4y + 4r^2s^3u^3v^2 - 10\,560r^2s^3uy^3 + 248r^2s^3v^2y^2 + 27r^2s^2t^4y^2 - 18r^2s^2t^3uvy + 4r^2s^2t^3v^3 + 4r^2s^2t^2u^3y - r^2s^2t^2u^2v^2 - \\
& 9720r^2s^2t^2y^3 + 10\,152r^2s^2tuvy^2 - 682r^2s^2tv^3y + 4816r^2s^2u^3y^2 - 5428r^2s^2u^2v^2y + 1020r^2s^2uv^4 - 43\,200r^2s^2y^4 + \\
& 3942r^2st^3vy^2 - 4536r^2st^2u^2y^2 - 2412r^2st^2uv^2y + 560r^2st^2v^4 + 3272r^2stu^3vy - 746r^2stu^2v^3 + 31\,320r^2stvy^3 - 576r^2su^5y + \\
& 144r^2su^4v^2 + 6480r^2su^2y^3 - 8748r^2suv^2y^2 + 1700r^2sv^4y + 162r^2t^4uy^2 - 108r^2t^3u^2vy + 24r^2t^3uv^3 + 24r^2t^2u^4y - \\
& 6r^2t^2u^3v^2 + 27\,540r^2t^2uy^3 - 15\,417r^2t^2v^2y^2 - 16\,632r^2tu^2vy^2 + 12\,330r^2tuv^3y - 2000r^2tv^5 + 192r^2u^4y^2 - 248r^2u^3v^2y + \\
& 50r^2u^2v^4 - 32\,400r^2uy^4 + 540r^2v^2y^3 + 6912rs^4ty^3 - 640rs^4uvy^2 + 144rs^4v^3y - 4464rs^3t^2vy^2 - 2496rs^3tu^2y^2 + 3272rs^3tuv^2y - \\
& 630rs^3tv^4 - 96rs^3u^3vy + 24rs^3u^2v^3 + 21\,888rs^3vy^3 + 2808rs^2t^3uy^2 - 108rs^2t^3v^2y - 1584rs^2t^2u^2vy + 356rs^2t^2uv^3 + \\
& 320rs^2tu^4y - 80rs^2tu^3v^2 + 3456rs^2tuy^3 - 16\,632rs^2tv^2y^2 - 15\,264rs^2u^2vy^2 + 13\,040rs^2uv^3y - 2250rs^2v^5 - 486rst^5y^2 +
\end{aligned}$$

$324rst^4uvy - 72rst^4v^3 - 72rst^3u^3y + 18rst^3u^2v^2 - 21\,384rst^3y^3 + 22\,896rst^2uvy^2 - 1980rst^2v^3y + 5760rstu^3y^2 -$
$10\,152rstu^2v^2y + 2050rstuv^4 - 77\,760rsty^4 + 640rsu^4vy - 160rsu^3v^3 + 31\,968rsuvy^3 - 1800rsv^3y^2 + 6318rt^4vy^2 -$
$5832rt^3u^2y^2 - 3942rt^3uv^2y + 900rt^3v^4 + 4464rt^2u^3vy - 1020rt^2u^2v^3 + 15\,552rt^2vy^3 - 768rtu^5y + 192rtu^4v^2 + 46\,656$
$rtu^2y^3 - 31\,320rtuv^2y^2 + 2250rtv^4y - 21\,888ru^3vy^2 + 15\,600ru^2v^3y - 2500ruv^5 - 38\,880rvy^4 - 1024s^6y^3 + 768s^5tvy^2 +$
$512s^5u^2y^2 - 576s^5uv^2y + 108s^5v^4 - 576s^4t^2uy^2 + 24s^4t^2v^2y + 320s^4tu^2vy - 72s^4tuv^3 - 64s^4u^4y + 16s^4u^3v^2 - 9216s^4uy^3 +$
$192s^4v^2y^2 + 108s^3t^4y^2 - 72s^3t^3uvy + 16s^3t^3v^3 + 16s^3t^2u^3y - 4s^3t^2u^2v^2 + 8640s^3t^2y^3 + 5760s^3tuvy^2 + 120s^3tv^3y +$
$4352s^3u^3y^2 - 4816s^3u^2v^2y + 900s^3uv^4 - 13\,824s^3y^4 - 5832s^2t^3vy^2 - 8208s^2t^2u^2y^2 + 4536s^2t^2uv^2y - 825s^2t^2v^4 +$
$2496s^2tu^3vy - 560s^2tu^2v^3 + 46\,656s^2tvy^3 - 512s^2u^5y + 128s^2u^4v^2 - 17\,280s^2u^2y^3 - 6480s^2uv^2y^2 + 1500s^2v^4y +$
$4860st^4uy^2 - 162st^4v^2y - 2808st^3u^2vy + 630st^3uv^3 + 576st^2u^4y - 144st^2u^3v^2 + 3888st^2uy^3 - 27\,540st^2v^2y^2 - 3456stu^2vy^2 +$
$19\,800stuv^3y - 3750stv^5 + 9216su^4y^2 - 10\,560su^3v^2y + 2000su^2v^4 - 62\,208suy^4 + 32\,400sv^2y^3 - 729t^6y^2 + 486t^5uvy -$
$108t^5v^3 - 108t^4u^3y + 27t^4u^2v^2 - 8748t^4y^3 + 21\,384t^3uvy^2 - 1350t^3v^3y - 8640t^2u^3y^2 - 9720t^2u^2v^2y + 2250t^2uv^4 -$
$34\,992t^2y^4 + 6912tu^4vy - 1600tu^3v^3 + 77\,760tuvy^3 - 27\,000tv^3y^2 - 1024u^6y + 256u^5v^2 + 13\,824u^3y^3 - 43\,200u^2v^2$
$y^2 + 22\,500uv^4y - 3125v^6 - 46\,656y^5$,

respectively.

3. Some Identities

In this section, we obtain some identities of (r,s,t,u,v,y), Lucas (r,s,t,u,v,y) and modified (r,s,t,u,v,y) numbers. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

LEMMA 7. *The following equalities are true:*

$$\begin{aligned} yH_n &= -vG_{n+2}+(6y+rv)G_{n+1}+(sv-5ry)G_n+(tv-4sy)G_{n-1}+(uv-3ty)G_{n-2}+(v^2-2uy)G_{n-3}, \quad (3.1)\\ H_n &= 6G_{n+1}-5rG_n-4sG_{n-1}-3tG_{n-2}-2uG_{n-3}-vG_{n-4}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (3.1). To show (3.1), writing

$$H_n = a\times G_{n+2}+b\times G_{n+1}+c\times G_n+d\times G_{n-1}+e\times G_{n-2}+f\times G_{n-3}$$

and solving the system of equations

$$\begin{aligned} H_0 &= a\times G_2+b\times G_1+c\times G_0+d\times G_{-1}+e\times G_{-2}+f\times G_{-3}\\ H_1 &= a\times G_3+b\times G_2+c\times G_1+d\times G_0+e\times G_{-1}+f\times G_{-2}\\ H_2 &= a\times G_4+b\times G_3+c\times G_2+d\times G_1+e\times G_0+f\times G_{-1}\\ H_3 &= a\times G_5+b\times G_4+c\times G_3+d\times G_2+e\times G_1+f\times G_0\\ H_4 &= a\times G_6+b\times G_5+c\times G_4+d\times G_3+e\times G_2+f\times G_1\\ H_5 &= a\times G_7+b\times G_6+c\times G_5+d\times G_4+e\times G_3+f\times G_2 \end{aligned}$$

we find that $a=\frac{1}{y^2}(v^2-2uy), b=-\frac{1}{y^2}(rv^2+yv-2ruy), c=\frac{1}{y^2}(-sv^2+rvy+6y^2+2suy), d=-\frac{1}{y^2}(tv^2-svy+5ry^2-2tuy), e=-\frac{1}{y^2}(-2u^2y+uv^2-tvy+4sy^2), f=-\frac{1}{y^2}(v^3-3uvy+3ty^2)$. The other equalitity can be proved similarly. $\square$

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

LEMMA 8. *The following equalities are true:*

(a): $yE_n = -G_{n+6} + rG_{n+5} + sG_{n+4} + tG_{n+3} + uG_{n+2} + (v+y)G_{n+1}$.

(b): $E_n = G_{n+1} - G_n$.

(c): $(r+s+t+u+v+y-1)G_n = E_{n+5} - (r-1)E_{n+4} - (r+s-1)E_{n+3} - (r+s+t-1)E_{n+2} - (r+s+t+u-1)E_{n+1} - (r+s+t+u+v-1)E_n$.

(d): $(r+s+t+u+v+y-1)G_n = E_{n+4} - (r-1)E_{n+3} - (r+s-1)E_{n+2} - (r+s+t-1)E_{n+1} - (r+s+t+u-1)E_n + yE_{n-1}$.

(e): $(r+s+t+u+v+y-1)G_n = E_{n+3} - (r-1)E_{n+2} - (r+s-1)E_{n+1} - (r+s+t-1)E_n + (v+y)E_{n-1} + yE_{n-2}$.

(f): $(r+s+t+u+v+y-1)G_n = E_{n+2} - (r-1)E_{n+1} - (r+s-1)E_n + (u+v+y)E_{n-1} + (v+y)E_{n-2} + yE_{n-3}$.

(g): $(r+s+t+u+v+y-1)G_n = E_{n+1} - (r-1)E_n + (t+u+v+y)E_{n-1} + (u+v+y)E_{n-2} + (v+y)E_{n-3} + yE_{n-4}$.

Note that all the identities in the above lemma can be proved by induction as well.

Thirdly, we give a basic relations between $\{H_n\}$ and $\{E_n\}$.

LEMMA 9. *The following equalitity is true:*

$(r+s+t+u+v+y-1)H_n = (6-4s-3t-2u-v-5r)E_{n+1} + (2s-5r+3t+4u+5v+6y+4rs+3rt+2ru+rv+5r^2)E_n + (3t-4s+4u+5v+6y+4rs-2rt-3ru+3st-4rv+2su+sv-5ry+4s^2)E_{n-1} + (4u-3t+5v+6y+3rt-3ru+3st-4rv-2su-3sv+2tu+tv-5ry-4sy+3t^2)E_{n-2} + (5v-2u+6y+2ru-4rv+2su-3sv+2tu-2tv-5ry+uv-4sy-3ty+2u^2)E_{n-3} + (6y-v+rv+sv+tv-5ry+uv-4sy-3ty-2uy+v^2)E_{n-4}$.

We now present a few special identities for the modified (r, s, t, u, v, y) sequence $\{E_n\}$.

THEOREM 10. *(Catalan's identity) For all integers n and m, the following identity holds*

$$E_{n+m}E_{n-m} - E_n^2 = G_{m+n+1}\left(G_{-m+n+1} - G_{-m+n}\right) + G_{m+n}\left(G_{-m+n} - G_{-m+n+1}\right) - \left(G_{n+1} - G_n\right)^2.$$

Proof. We use the identity

$$E_n = G_{n+1} - G_n. \quad \square$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the modified (r, s, t, u, v) sequnce.

COROLLARY 11. *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+2} - G_{n+1})(G_n - G_{n-1}) - (G_{n+1} - G_n)^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using $E_n = G_{n+1} - G_n$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of modified (r, s, t, u, v, y) sequence $\{E_n\}$.

THEOREM 12. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = G_{m+2}\left(G_{n+1} - G_n\right) + G_{m+1}\left(G_n - G_{n+2}\right) + G_m\left(G_{n+2} - G_{n+1}\right).$$

(b): *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+3} - G_{n+2})(G_{n+2} - G_{n+1})(G_n - G_{n-1})(G_{n-1} - G_{n-2}) - (G_{n+1} - G_n)^4.$$

(c): *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+2} - G_{n+1})(G_{n+3} - G_{n+2})(G_{n+7} - G_{n+6}) - (G_{n+4} - G_{n+3})^3.$$

Proof. Use the identity $E_n = G_{n+1} - G_n$. □

4. Linear Sums

The following theorem presents sum formulas of generalized (r,s,t,u,v,y) numbers.

THEOREM 13. *For all integers m and j, we have*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{\Lambda + \Psi_1}{\Omega} \tag{4.1}$$

where

$\Lambda = -24W_{mn+5m+j} + 24(H_m - 1)W_{mn+4m+j} - 12(H_m^2 - 2H_m - H_{2m} + 2)W_{mn+3m+j} + 4(H_m^3 - 3H_m^2 - 3H_{2m}H_m + 3H_{2m} + 2H_{3m} + 6H_m - 6)W_{mn+2m+j} + (-H_m^4 + 4H_m^3 + 6H_m^2 H_{2m} - 12H_m^2 - 3H_{2m}^2 - 12H_{2m}H_m - 8H_{3m}H_m + 6H_{4m} + 8H_{3m} + 12H_{2m} + 24H_m - 24)W_{mn+m+j} + 24\,(-y)^m\, W_{mn+j}$

$\Psi_1 = 24W_{5m+j} - 24(H_m - 1)W_{4m+j} + 12(H_m^2 - 2H_m - H_{2m} + 2)W_{3m+j} - 4(H_m^3 - 3H_m^2 - 3H_{2m}H_m + 3H_{2m} + 2H_{3m} + 6H_m - 6)W_{2m+j} - (-H_m^4 + 4H_m^3 + 6H_m^2 H_{2m} - 12H_m^2 - 3H_{2m}^2 - 12H_{2m}H_m - 8H_{3m}H_m + 6H_{4m} + 8H_{3m} + 12H_{2m} + 24H_m - 24)W_{m+j} - (-H_m^4 + 4H_m^3 - 12H_m^2 + 6H_m^2 H_{2m} - 3H_{2m}^2 - 8H_{3m}H_m - 12H_{2m}H_m + 6H_{4m} + 8H_{3m} + 12H_{2m} + 24H_m + 24\,(-y)^m\, H_{-m} - 24)W_j$

$\Omega = H_m^4 - 4H_m^3 + 3H_{2m}^2 + 12H_m^2 - 6H_m^2 H_{2m} + 8H_{3m}H_m + 12H_{2m}H_m - 6H_{4m} - 8H_{3m} - 12H_{2m} - 24H_m - 24\,(-y)^m\,(H_{-m} - 1) + 24$

Proof. Note that

$$\begin{aligned}
\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j} \\
&\quad + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j} + A_4\delta^{mk+j} + A_5\lambda^{mk+j} + A_6\mu^{mk+j}) \\
&= W_{mn+j} + A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + A_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + A_3\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right) \\
&\quad + A_4\delta^j\left(\frac{\delta^{mn}-1}{\delta^m-1}\right) + A_5\lambda^j\left(\frac{\lambda^{mn}-1}{\lambda^m-1}\right) + A_6\mu^j\left(\frac{\mu^{mn}-1}{\mu^m-1}\right).
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (4.1) as required. □

Note that (4.1) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{\Lambda + \Psi_2}{\Omega}$$

where

$\Psi_2 = 24W_{5m+j} - 24(H_m - 1)W_{4m+j} + 12(H_m^2 - 2H_m - H_{2m} + 2)W_{3m+j} - 4(H_m^3 - 3H_m^2 - 3H_{2m}H_m + 3H_{2m} + 2H_{3m} + 6H_m - 6)W_{2m+j} - (-H_m^4 + 4H_m^3 + 6H_m^2 H_{2m} - 12H_m^2 - 3H_{2m}^2 - 12H_{2m}H_m - 8H_{3m}H_m + 6H_{4m} + 8H_{3m} + 12H_{2m} + 24H_m - 24)W_{m+j} - 24\,(-y)^m\, W_j$.

As special cases of the above theorem, we have the following corollaries. Firstly, as special cases of the above theorem, we have the following corollary for the generalized Hexanacci numbers.

COROLLARY 14. *The following identities hold:*

(1) $m = 1,\ j = 0$.

(a): $\sum_{k=0}^{n} H_k = \frac{1}{5}(H_{n+5} - H_{n+3} - 2H_{n+2} - 3H_{n+1} + H_n - 1)$.

(b): $\sum_{k=0}^{n} E_k = \frac{1}{5}(E_{n+5} - E_{n+3} - 2E_{n+2} - 3E_{n+1} + E_n + 9)$.

(2) $m = -1,\ j = 0$.

(a): $\sum_{k=0}^{n} H_{-k} = \frac{1}{5}(-H_{-n} - 5H_{-n-1} - 4H_{-n-2} - 3H_{-n-3} - 2H_{-n-4} - H_{-n-5} + 1)$.

(b): $\sum_{k=0}^{n} E_{-k} = \frac{1}{5}(-E_{-n} - 5E_{-n-1} - 4E_{-n-2} - 3E_{-n-3} - 2E_{-n-4} - E_{-n-5} + 21)$.

(3) $m = 3,\ j = -6$.

(a): $\sum_{k=0}^{n} H_{3k-6} = \frac{1}{5}(H_{3n+9} - 6H_{3n+6} - 13H_{3n+3} - 8H_{3n} - 3H_{3n-3} + H_{3n-6} - 7)$.

(b): $\sum_{k=0}^{n} E_{3k-6} = \frac{1}{5}(E_{3n+9} - 6E_{3n+6} - 13E_{3n+3} - 8E_{3n} - 3E_{3n-3} + E_{3n-6} + 83)$.

(4) $m = -3,\ j = 2$.

(a): $\sum_{k=0}^{n} H_{-3k+2} = \frac{1}{5}(-H_{-3n+2} + H_{-3n-1} + 8H_{-3n-4} + 3H_{-3n-7} - 2H_{-3n-10} - H_{-3n-13} + 7)$.

(b): $\sum_{k=0}^{n} E_{-3k+2} = \frac{1}{5}(-E_{-3n+2} + E_{-3n-1} + 8E_{-3n-4} + 3E_{-3n-7} - 2E_{-3n-10} - E_{-3n-13} + 22)$.

Secondly, as special cases of the above theorem, we have the following corollary for the generalized sixth-order Pell numbers.

Corollary 15. *The following identities hold:*

(1) $m = 1,\ j = 0$.

(a): $\sum_{k=0}^{n} P_k = \frac{1}{6}(P_{n+5} - P_{n+4} - 2P_{n+3} - 3P_{n+2} - 4P_{n+1} + P_n - 1)$.

(b): $\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+5} - Q_{n+4} - 2Q_{n+3} - 3Q_{n+2} - 4Q_{n+1} + Q_n + 14)$.

(c): $\sum_{k=0}^{n} E_k = \frac{1}{6}(E_{n+5} - E_{n+4} - 2E_{n+3} - 3E_{n+2} - 4E_{n+1} + E_n)$.

(2) $m = -1,\ j = 0$.

(a): $\sum_{k=0}^{n} P_{-k} = \frac{1}{6}(-P_{-n} - 5P_{-n-1} - 4P_{-n-2} - 3P_{-n-3} - 2P_{-n-4} - P_{-n-5} + 1)$.

(b): $\sum_{k=0}^{n} Q_{-k} = \frac{1}{6}(-Q_{-n} - 5Q_{-n-1} - 4Q_{-n-2} - 3Q_{-n-3} - 2Q_{-n-4} - Q_{-n-5} + 22)$.

(c): $\sum_{k=0}^{n} E_{-k} = \frac{1}{6}(-E_{-n} - 5E_{-n-1} - 4E_{-n-2} - 3E_{-n-3} - 2E_{-n-4} - E_{-n-5})$.

(3) $m = 3,\ j = -6$.

(a): $\sum_{k=0}^{n} P_{3k-6} = \frac{1}{18}(P_{3n+9} - 16P_{3n+6} - 32P_{3n+3} - 24P_{3n} - 16P_{3n-3} + P_{3n-6} - 25)$.

(b): $\sum_{k=0}^{n} Q_{3k-6} = \frac{1}{18}(Q_{3n+9} - 16Q_{3n+6} - 32Q_{3n+3} - 24Q_{3n} - 16Q_{3n-3} + Q_{3n-6} + 392)$.

(c): $\sum_{k=0}^{n} E_{3k-6} = \frac{1}{18}(E_{3n+9} - 16E_{3n+6} - 32E_{3n+3} - 24E_{3n} - 16E_{3n-3} + E_{3n-6} - 24)$.

(4) $m = -3,\ j = 2$.

(a): $\sum_{k=0}^{n} P_{-3k+2} = \frac{1}{18}(-P_{-3n+2} - 2P_{-3n-1} + 14P_{-3n-4} + 6P_{-3n-7} - 2P_{-3n-10} - P_{-3n-13} + 37)$.

(b): $\sum_{k=0}^{n} Q_{-3k+2} = \frac{1}{18}(-Q_{-3n+2} + 14Q_{-3n-4} - 2Q_{-3n-1} + 6Q_{-3n-7} - 2Q_{-3n-10} - Q_{-3n-13} + 142)$.

(c): $\sum_{k=0}^{n} E_{-3k+2} = \frac{1}{18}(-E_{-3n+2} - 2E_{-3n-1} + 14E_{-3n-4} + 6E_{-3n-7} - 2E_{-3n-10} - E_{-3n-13} + 24)$.

5. Matrices Related with Generalized (r,s,t,u,v,y) Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v & y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{5.1}$$

For matrix formulation (5.1), see [2]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \\ W_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r & s & t & u & v & y \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix}.$$

We define the square matrix A of order 6 as:

$$A = \begin{pmatrix} r & s & t & u & v & y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -y$. From (1.1) we have

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v & y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix}. \tag{5.2}$$

and from (5.1) (or using (5.2) and induction) we have

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v & y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (5.2) we have

$$\begin{pmatrix} G_{n+5} \\ G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v & y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{5.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=3}^{6} r_k G_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}G_{n+3-k} & r_6 G_n \\ G_n & \sum\limits_{k=4}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=4}^{6} r_k G_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}G_{n+3-k} & r_6 G_{n-1} \\ G_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}G_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=5}^{6} r_k G_{n+3-k} & r_6 G_{n-2} \\ G_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}G_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}G_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}G_{n+3-k} & r_6 G_{n-3} \\ G_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}G_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}G_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}G_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}G_{n+3-k} & r_6 G_{n-4} \\ G_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}G_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}G_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}G_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}G_{n+3-k} & r_6 G_{n-5} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}W_{n+3-k} & \sum\limits_{k=3}^{6} r_k W_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}W_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}W_{n+3-k} & r_6 W_n \\ W_n & \sum\limits_{k=4}^{8} r_{k-2}W_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}W_{n+3-k} & \sum\limits_{k=4}^{6} r_k W_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}W_{n+3-k} & r_6 W_{n-1} \\ W_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}W_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}W_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}W_{n+3-k} & \sum\limits_{k=5}^{6} r_k W_{n+3-k} & r_6 W_{n-2} \\ W_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}W_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}W_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}W_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}W_{n+3-k} & r_6 W_{n-3} \\ W_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}W_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}W_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}W_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}W_{n+3-k} & r_6 W_{n-4} \\ W_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}W_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}W_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}W_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}W_{n+3-k} & r_6 W_{n-5} \end{pmatrix}$$

where

$$r_1 = r, r_2 = s, r_3 = t, r_4 = u, r_5 = v, r_6 = y.$$

THEOREM 16. *For all integers $m, n \geq 0$, we have*

- **(a):** $B_n = A^n$.
- **(b):** $C_1 A^n = A^n C_1$.
- **(c):** $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

- **(a):** By expanding the vectors on the both sides of (5.3) to 6-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n. \ \square$$

Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2} + tA^{n-3} + uA^{n-4} + vA^{n-5} + yA^{n-6}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = (-y)^n$$

for all integers m and n.

THEOREM 17. *For $m, n \geq 0$, we have*

$$\begin{aligned} W_{n+m} &= W_nG_{m+1} + \sum_{i=1}^{6-1} W_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i}G_{m+1-j}\right) \qquad (5.4)\\ &= W_nG_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3} + yG_{m-4}) \\ &\quad + W_{n-2}(tG_m + uG_{m-1} + vG_{m-2} + yG_{m-3}) \\ &\quad + W_{n-3}(uG_m + vG_{m-1} + yG_{m-2}) + W_{n-4}(vG_m + yG_{m-1}) + yG_mW_{n-5} \end{aligned}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof. $\square$

REMARK 18. *By induction, it can be proved that for all integers $m, n \leq 0$, (5.4) holds. So for all integers m, n, (5.4) is true.*

Corollary 19. *For all integers m, n, we have*

$$\begin{aligned}
G_{n+m} &= G_nG_{m+1} + G_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3} + yG_{m-4}) \\
&\quad + G_{n-2}(tG_m + uG_{m-1} + vG_{m-2} + yG_{m-3}) \\
&\quad + G_{n-3}(uG_m + vG_{m-1} + yG_{m-2}) + G_{n-4}(vG_m + yG_{m-1}) + yG_mG_{n-5}, \\
H_{n+m} &= H_nG_{m+1} + H_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3} + yG_{m-4}) \\
&\quad + H_{n-2}(tG_m + uG_{m-1} + vG_{m-2} + yG_{m-3}) \\
&\quad + H_{n-3}(uG_m + vG_{m-1} + yG_{m-2}) + H_{n-4}(vG_m + yG_{m-1}) + yG_mH_{n-5}, \\
E_{n+m} &= E_nG_{m+1} + E_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3} + yG_{m-4}) \\
&\quad + E_{n-2}(tG_m + uG_{m-1} + vG_{m-2} + yG_{m-3}) \\
&\quad + E_{n-3}(uG_m + vG_{m-1} + yG_{m-2}) + E_{n-4}(vG_m + yG_{m-1}) + yG_mE_{n-5}.
\end{aligned}$$

or, in short formulas,

$$\begin{aligned}
G_{n+m} &= G_nG_{m+1} + \sum_{i=1}^{6-1} G_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i}G_{m+1-j}\right), \\
H_{n+m} &= H_nG_{m+1} + \sum_{i=1}^{6-1} H_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i}G_{m+1-j}\right), \\
E_{n+m} &= E_nG_{m+1} + \sum_{i=1}^{6-1} E_{n-i}\left(\sum_{j=1}^{6-i} r_{j+i}G_{m+1-j}\right).
\end{aligned}$$

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