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Generalized Leonardo Numbers

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Abstract

In this paper, we investigate the generalized Leonardo sequences and we deal with, in detail, three special cases, namely, modified Leonardo, Leonardo-Lucas and Leonardo sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between modified Leonardo, Leonardo-Lucas, Leonardo numbers and Fibonacci, Lucas numbers.

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Leonardo numbers, Leonardo-Lucas numbers, Tribonacci numbers, Fibonacci numbers, Lucas numbers.

1. Introduction

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are

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defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \ F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \ L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

In [5], Catarino and Borges introduced a new sequence of numbers called Lenonardo numbers. They defined Lenonardo numbers as

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with $l_0 = 1, l_1 = 1$. The first few values of Lenonardo numbers are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, \ldots$$

Lenonardo sequences has been studied by many authors, see for example, [1,4,14,24].

Now, we define two sequences related to Lenonardo sequence. Modified Leonardo and Leonardo-Lucas numbers are defined as

$$G_n = G_{n-1} + G_{n-2} + 1, \qquad \textbf{with} \ \ G_0 = 0, G_1 = 1, \quad n \geq 2,$$

and

$$H_n = H_{n-1} + H_{n-2} - 1, \qquad \textbf{with} \ \ H_0 = 3, H_1 = 2, \qquad n \geq 2,$$

respectively. The first few values of modified Leonardo and Leonardo-Lucas numbers are

$$0, 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, \ldots$$

and

$$3, 2, 4, 5, 8, 12, 19, 30, 48, 77, 124, 200, 323, \ldots$$

respectively. The sequences $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ satisfy the following third order linear recurences:

$$\begin{aligned} G_n &= 2G_{n-1} - G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 2, \\ H_n &= 2H_{n-1} - H_{n-3}, \qquad H_0 = 3, H_1 = 2, H_2 = 4, \\ l_n &= 2l_{n-1} - l_{n-3}, \qquad l_0 = 1, l_1 = 1, l_2 = 3. \end{aligned}$$

There are close relations between modified Leonardo, Leonardo-Lucas, Leonardo numbers and Fibonacci, Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} G_n &= F_{n+2}-1, \\ H_n &= L_n+1, \\ l_n &= 2F_{n+1}-1, \end{aligned}$$

and

$$\begin{aligned} 5G_n &= 3L_{n+1}+L_n-5, \\ H_n &= 2F_{n+1}-F_n+1, \\ 5l_n &= 2L_{n+1}+4L_n-5. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., modified Leonardo, Leonardo-Lucas, Leonardo numbers). First, we recall some properties of generalized Tribonacci numbers. The generalized (r,s,t) **sequence (or generalized Tribonacci sequence or generalized** 3**-step Fibonacci sequence)**

$$\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$**) is defined as follows:**

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}, \qquad W_0=a, W_1=b, W_2=c, \quad n\geq 3 \tag{1.1}$$

where W_0,W_1,W_2 **are arbitrary complex (or real) numbers and** r,s,t **are real numbers.**

This sequence has been studied by many authors, see for example [2,3,6,7,8,10,11,12,13,17,18,23,25,26].

The sequence $\{W_n\}_{n\geq 0}$ **can be extended to negative subscripts by defining**

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n=1,2,3,...$ **when** $t\neq 0$**. Therefore, recurrence (1.1) holds for all integer** n.

As $\{W_n\}$ **is a third-order recurrence sequence (difference equation), it's characteristic equation is**

$$x^3-rx^2-sx-t=0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3}+A+B, \\ \beta &= \beta(r,s,t) = \frac{r}{3}+\omega A+\omega^2 B, \\ \gamma &= \gamma(r,s,t) = \frac{r}{3}+\omega^2 A+\omega B, \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3},\ B=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3}, \\ \Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4},\quad \omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3). \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha+\beta+\gamma &= r, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r,s,t)>0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that the generalized (r,s,t) numbers (the generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n=\frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.3}$$

where

$$p_1=W_2-(\beta+\gamma)W_1+\beta\gamma W_0,\ p_2=W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,\ p_3=W_2-(\alpha+\beta)W_1+\alpha\beta W_0.$$

(1.3) can be written in the following form:

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n$$

where

$$A_1=\frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)},A_2=\frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)},A_3=\frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty}W_nx^n$ of the sequence W_n.

Lemma 1. *Suppose that $f_{W_n}(x)=\sum_{n=0}^{\infty}W_nx^n$ is the ordinary generating function of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty}W_nx^n$ is given by*

$$\sum_{n=0}^{\infty}W_nx^n=\frac{W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2}{1-rx-sx^2-tx^3}. \tag{1.4}$$

We next find Binet's formula of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}$ by the use of generating function for W_n.

Theorem 2. *(Binet's formula of the generalized (r,s,t) numbers (the generalized Tribonacci numbers)) For all integers n, we have*

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.5}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ q_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ q_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Note that from (1.3) and (1.5) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{1.6}$$

For matrix formulation (1.6), see [9]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

Now, we present Simson's formula of generalized Tribonacci numbers.

Theorem 3 (Simson's Formula of Generalized Tribonacci Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{1.7}$$

Proof. For a proof, see Soykan [16]. □

Next, we consider two special cases of the generalized (r,s,t) sequence $\{W_n\}$ which we call them (r,s,t) and Lucas (r,s,t) sequences. (r,s,t) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t) sequence

$\{H_n\}_{n\geq 0}$ **are defined, respectively, by the third-order recurrence relations**

$$G_{n+3} = rG_{n+2} + sG_{n+1} + tG_n, \qquad G_0 = 0, G_1 = 1, G_2 = r, \tag{1.8}$$

$$H_{n+3} = rH_{n+2} + sH_{n+1} + tH_n, \qquad H_0 = 3, H_1 = r, H_2 = 2s + r^2. \tag{1.9}$$

The sequences $\{G_n\}_{n\geq 0}$ **and** $\{H_n\}_{n\geq 0}$ **can be extended to negative subscripts by defining**

$$G_{-n} = -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)},$$

$$H_{-n} = -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}$$

for $n = 1, 2, 3, ...$ **respectively. Therefore, recurrences (1.8)-(1.9) hold for all integers** n.

For all integers n, (r, s, t)**, Lucas** (r, s, t) **and modified** (r, s, t) **numbers (using initial conditions in (1.3) or (1.5)) can be expressed using Binet's formulas as**

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively.

Lemma 1 gives the following results as particular examples (generating functions of (r, s, t) **and Lucas** (r, s, t) **numbers).**

Corollary 4. *Generating functions of* (r, s, t)*, Lucas* (r, s, t) *and modified* (r, s, t) *numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1 - rx - sx^2 - tx^3}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{3 - 2rx - sx^2}{1 - rx - sx^2 - tx^3}, \end{aligned}$$

respectively.

The following theorem shows that the generalized Tribonacci sequence W_n **at negative indices can be expressed by the sequence itself at positive indices.**

Theorem 5. *For* $n \in \mathbb{Z}$*, we have*

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. For the proof, see Soykan [19, Theorem 2.]. □

Using Theorem 5, we have the following corollary, see Soykan [19, Corollary 6].

Corollary 6. *For* $n \in \mathbb{Z}$*, we have*

(a) $G_{-n} = \frac{1}{t^{n+1}}((2rt - s^2)G_n^2 + tG_{2n} + sG_{n+2}G_n - (3t + rs)G_{n+1}G_n)$.

(b) $H_{-n} = \frac{1}{2t^n}(H_n^2 - H_{2n})$.

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 3$ in Theorem 5,

$$\begin{aligned} G_{-n} &= t^{-n}(G_{2n} - H_nG_n + \frac{1}{2}(H_n^2 - H_{2n})G_0) = t^{-n}(G_{2n} - H_nG_n), \\ H_{-n} &= t^{-n}(H_{2n} - H_nH_n + \frac{1}{2}(H_n^2 - H_{2n})H_0) = \frac{1}{2t^n}(H_n^2 - H_{2n}), \end{aligned}$$

respectively.

2 Generalized Leonardo Sequence

In this paper, we consider the case $r = 2, s = 0, t = -1$ and in this case we write $V_n = W_n$. A generalized Leonardo sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} - V_{n-3} \tag{2.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = 2V_{-(n-2)} - V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized Leonardo numbers. Binet formula of generalized Leonardo numbers can be given as

$$\begin{aligned} V_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \qquad (2.2) \\ &= \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{(\alpha-\beta)} - z_3 \end{aligned}$$

where

$$\begin{aligned} z_1 &= V_2 - (2-\alpha)V_1 + (1-\alpha)V_0, && (2.3) \\ z_2 &= V_2 - (2-\beta)V_1 + (1-\beta)V_0, && (2.4) \\ z_3 &= V_2 - V_1 - V_0. && (2.5) \end{aligned}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 + 1 = \left(x^2 - x - 1\right)(x - 1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= \frac{1+\sqrt{5}}{2}, \\ \beta &= \frac{1-\sqrt{5}}{2}, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 2, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= 0, \\ \alpha\beta\gamma &= -1, \end{aligned}$$

or

$$\alpha+\beta=1,\ \alpha\beta=-1.$$

The first few generalized Leonardo numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Leonardo numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$2V_1 - V_2$
2	V_2	$2V_0 - V_1$
3	$2V_2 - V_0$	$4V_1 - V_0 - 2V_2$
4	$4V_2 - V_1 - 2V_0$	$4V_0 - 4V_1 + V_2$
5	$7V_2 - 2V_1 - 4V_0$	$9V_1 - 4V_0 - 4V_2$
6	$12V_2 - 4V_1 - 7V_0$	$9V_0 - 12V_1 + 4V_2$
7	$20V_2 - 7V_1 - 12V_0$	$22V_1 - 12V_0 - 9V_2$
8	$33V_2 - 12V_1 - 20V_0$	$22V_0 - 33V_1 + 12V_2$
9	$54V_2 - 20V_1 - 33V_0$	$56V_1 - 33V_0 - 22V_2$
10	$88V_2 - 33V_1 - 54V_0$	$56V_0 - 88V_1 + 33V_2$
11	$143V_2 - 54V_1 - 88V_0$	$145V_1 - 88V_0 - 56V_2$
12	$232V_2 - 88V_1 - 143V_0$	$145V_0 - 232V_1 + 88V_2$
13	$376V_2 - 143V_1 - 232V_0$	$378V_1 - 232V_0 - 145V_2$

Now we define three special cases of the sequence $\{V_n\}$. Modified Leonardo sequence $\{G_n\}_{n\geq 0}$, Leonardo-Lucas sequence $\{H_n\}_{n\geq 0}$ and Leonardo sequence $\{l_n\}_{n\geq 0}$ are defined, re-

spectively, by the third-order recurrence relations

$$
\begin{aligned}
G_n &= 2G_{n-1} - G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 2, && (2.6)\\
H_n &= 2H_{n-1} - H_{n-3}, \qquad H_0 = 3, H_1 = 2, H_2 = 4, && (2.7)\\
l_n &= 2l_{n-1} - l_{n-3}, \qquad l_0 = 1, l_1 = 1, l_2 = 3, && (2.8)
\end{aligned}
$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{l_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
G_{-n} &= 2G_{-(n-2)} - G_{-(n-3)}\\
H_{-n} &= 2H_{-(n-2)} - H_{-(n-3)}\\
l_{-n} &= 2l_{-(n-2)} - l_{-(n-3)}
\end{aligned}
$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.6)-(2.8) hold for all integer n.

G_n, H_n and l_n are the sequences A000071, A001612, A001595 in [15], respectively.

Next, we present the first few values of the modified Leonardo, Leonardo-Lucas and Leonardo numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
G_n	0	1	2	4	7	12	20	33	54	88	143	232	376	609
G_{-n}		0	-1	0	-2	1	-4	4	-9	12	-22	33	-56	88
H_n	3	2	4	5	8	12	19	30	48	77	124	200	323	522
H_{-n}		0	4	-3	8	-10	19	-28	48	-75	124	-198	323	-520
l_{-n}	1	1	3	5	9	15	25	41	67	109	177	287	465	753
l_{-n}		-1	1	-3	3	-7	9	-17	25	-43	67	-111	177	-289

For all integers n, modified Leonardo, Leonardo-Lucas and Leonardo numbers (using initial conditions in (2.3)) can be expressed using Binet's formulas as

$$
\begin{aligned}
G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} = \frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} - 1\\
H_n &= \alpha^n + \beta^n + \gamma^n = \alpha^n + \beta^n + 1\\
l_n &= \frac{2(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} - 1
\end{aligned}
$$

respectively. Note that Binet's formulas of Fibonacci and Lucas numbers, respectively, are

$$
\begin{aligned}
F_n &= \frac{\alpha^n}{\alpha-\beta} + \frac{\beta^n}{\beta-\alpha} = \frac{\alpha^n-\beta^n}{\alpha-\beta},\\
L_n &= \alpha^n + \beta^n,
\end{aligned}
$$

and so

$$G_n = F_{n+2} - 1, \quad (2.9)$$

$$H_n = L_n + 1, \quad (2.10)$$

$$l_n = 2F_{n+1} - 1. \quad (2.11)$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ **of the sequence** V_n.

Lemma 7. *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized Leonardo sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1)x^2}{1 - 2x + x^3}.$$

Proof. Take $r = 2, s = 0, t = -1$ **in Lemma 1.** $\square$

The previous lemma gives the following results as particular examples.

Corollary 8. *Generated functions of modified Leonardo, Leonardo-Lucas and Leonardo numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1 - 2x + x^3}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{3 - 4x}{1 - 2x + x^3}, \\
\sum_{n=0}^{\infty} l_n x^n &= \frac{1 - x + x^2}{1 - 2x + x^3},
\end{aligned}$$

respectively.

3 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$**, namely,**

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Leonardo sequence $\{V_n\}_{n\geq 0}$.

Theorem 9 (Simson Formula of Generalized Leonardo Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = (-1)^n(-V_2^3 + V_1^3 - V_0^3 + 2(2V_1V_2^2 + V_0V_1^2 - 2V_1^2V_2 + V_0^2V_2) - 3V_0V_1V_2).$$

Proof. Take $r = 2, s = 0, t = -1$ in Theorem 3. □

The previous theorem gives the following results as particular examples.

Corollary 10. *For all integers n, Simson formula of modified Leonardo, Leonardo-Lucas and Leonardo numbers are given as*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = (-1)^n,$$

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = 5(-1)^n,$$

$$\begin{vmatrix} l_{n+2} & l_{n+1} & l_n \\ l_{n+1} & l_n & l_{n-1} \\ l_n & l_{n-1} & l_{n-2} \end{vmatrix} = 4(-1)^{n+1},$$

respectively.

4 Some Identities

In this section, we obtain some identities of modified Leonardo, Leonardo-Lucas and Leonardo numbers. First, we can give a few basic relations between $\{V_n\}$ and $\{G_n\}$.

Lemma 11. *The following equalities are true:*

(a) $V_n = (4V_1 - V_0 - 2V_2)G_{n+4} + (4V_0 - 9V_1 + 4V_2)G_{n+3} + (4V_1 - 4V_0 - V_2)G_{n+2}.$

(b) $V_n = (2V_0 - V_1)G_{n+3} + (4V_1 - 4V_0 - V_2)G_{n+2} + (V_0 - 4V_1 + 2V_2)G_{n+1}.$

(c) $V_n = (2V_1 - V_2)G_{n+2} + (V_0 - 4V_1 + 2V_2)G_{n+1} + (V_1 - 2V_0)G_n.$

(d) $V_n = V_0G_{n+1} + (V_1 - 2V_0)G_n + (V_2 - 2V_1)G_{n-1}.$

(e) $V_n = V_1G_n + (V_2 - 2V_1)G_{n-1} - V_0G_{n-2}.$

(f) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n = -(V_2^2+V_0V_1-2V_1V_2)V_{n+4}+(V_1^2+2V_2^2+2V_0V_1-V_0V_2-4V_1V_2)V_{n+3}-(V_0^2-2V_0V_2+V_1V_2)V_{n+2}$.

(g) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n = (V_1^2-V_0V_2)V_{n+3}-(V_0^2-2V_0V_2+V_1V_2)V_{n+2}+(V_2^2+V_0V_1-2V_1V_2)V_{n+1}$.

(h) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n = -(V_0^2-2V_1^2+V_1V_2)V_{n+2}+(V_2^2+V_0V_1-2V_1V_2)V_{n+1}+(-V_1^2+V_0V_2)V_n$.

(i) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n = -(2V_0^2-4V_1^2-V_2^2-V_0V_1+4V_1V_2)V_{n+1}+(-V_1^2+V_0V_2)V_n+(V_0^2-2V_1^2+V_1V_2)V_{n-1}$.

(j) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n = -(4V_0^2-7V_1^2-2V_2^2-2V_0V_1-V_0V_2+8V_1V_2)V_n+(V_0^2-2V_1^2+V_1V_2)V_{n-1}+(2V_0^2-4V_1^2-V_2^2-V_0V_1+4V_1V_2)V_{n-2}$.

Proof. Note that all the identities hold for all integers n. **We prove (a). To show (a), writing**

$$V_n = a\times G_{n+4}+b\times G_{n+3}+c\times G_{n+2}$$

and solving the system of equations

$$\begin{aligned} V_0 &= a\times G_4+b\times G_3+c\times G_2 \\ V_1 &= a\times G_5+b\times G_4+c\times G_3 \\ V_2 &= a\times G_6+b\times G_5+c\times G_4 \end{aligned}$$

we find that $a=4V_1-V_0-2V_2, b=4V_0-9V_1+4V_2, c=4V_1-4V_0-V_2$. **The other equalities can be proved similarly.**

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{V_n\}$ **and** $\{H_n\}$.

Lemma 12. *The following equalities are true:*

(a) $5V_n = -(4V_0+8V_1-7V_2)H_{n+4}+(V_0+12V_1-8V_2)H_{n+3}+(8V_0+V_1-4V_2)H_{n+2}$.

(b) $5V_n = -(7V_0+4V_1-6V_2)H_{n+3}+(8V_0+V_1-4V_2)H_{n+2}+(4V_0+8V_1-7V_2)H_{n+1}$.

(c) $5V_n = -(6V_0+7V_1-8V_2)H_{n+2}+(4V_0+8V_1-7V_2)H_{n+1}+(7V_0+4V_1-6V_2)H_n$.

(d) $5V_n = -(8V_0+6V_1-9V_2)H_{n+1}+(7V_0+4V_1-6V_2)H_n+(6V_0+7V_1-8V_2)H_{n-1}$.

(e) $5V_n = -(9V_0+8V_1-12V_2)H_n+(6V_0+7V_1-8V_2)H_{n-1}+(8V_0+6V_1-9V_2)H_{n-2}$.

(f) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)H_n = (3V_1^2+4V_2^2+4V_0V_1-3V_0V_2-8V_1V_2)V_{n+4}-(3V_0^2+4V_1^2+8V_2^2+8V_0V_1-10V_0V_2-13V_1V_2)V_{n+3}+(4V_0^2+3V_2^2+3V_0V_1-8V_0V_2-2V_1V_2)V_{n+2}$.

(g) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)H_n = -(3V_0^2-2V_1^2-4V_0V_2+3V_1V_2)V_{n+3}+(4V_0^2+3V_2^2+3V_0V_1-8V_0V_2-2V_1V_2)V_{n+2}-(3V_1^2+4V_2^2+4V_0V_1-3V_0V_2-8V_1V_2)V_{n+1}$.

(h) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)H_n = -(2V_0^2-4V_1^2-3V_2^2-3V_0V_1+8V_1V_2)V_{n+2}-(3V_1^2+4V_2^2+4V_0V_1-3V_0V_2-8V_1V_2)V_{n+1}+(3V_0^2-2V_1^2-4V_0V_2+3V_1V_2)V_n$.

(i) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)H_n = -(4V_0^2-5V_1^2-2V_2^2-2V_0V_1-3V_0V_2+8V_1V_2)V_{n+1}+(3V_0^2-2V_1^2-4V_0V_2+3V_1V_2)V_n+(2V_0^2-4V_1^2-3V_2^2-3V_0V_1+8V_1V_2)V_{n-1}$.

(j) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)H_n = -(5V_0^2-8V_1^2-4V_2^2-4V_0V_1-2V_0V_2+13V_1V_2)V_n+(2V_0^2-4V_1^2-3V_2^2-3V_0V_1+8V_1V_2)V_{n-1}+(4V_0^2-5V_1^2-2V_2^2-2V_0V_1-3V_0V_2+8V_1V_2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{l_n\}$.

Lemma 13. *The following equalities are true:*

(a) $2V_n = (3V_0-V_2)l_{n+4}-(5V_0-3V_1)l_{n+3}-(5V_1-3V_2)l_{n+2}$.

(b) $2V_n = (V_0+3V_1-2V_2)l_{n+3}-(5V_1-3V_2)l_{n+2}-(3V_0-V_2)l_{n+1}$.

(c) $2V_n = (2V_0+V_1-V_2)l_{n+2}-(3V_0-V_2)l_{n+1}-(V_0+3V_1-2V_2)l_n$.

(d) $2V_n = (V_0+2V_1-V_2)l_{n+1}-(V_0+3V_1-2V_2)l_n-(2V_0+V_1-V_2)l_{n-1}$.

(e) $2V_n = (V_0+V_1)l_n-(2V_0+V_1-V_2)l_{n-1}-(V_0+2V_1-V_2)l_{n-2}$.

(f) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)l_n = (V_0^2+V_1^2+V_2^2+V_0V_1-3V_0V_2-V_1V_2)V_{n+4}-(3V_0^2+V_1^2+3V_2^2+3V_0V_1-7V_0V_2-3V_1V_2)V_{n+3}+(V_0^2+V_1^2+3V_2^2+3V_0V_1-3V_0V_2-5V_1V_2)V_{n+2}$.

(g) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)l_n = -(V_0^2-V_1^2+V_2^2+V_0V_1-V_0V_2-V_1V_2)V_{n+3}+(V_0^2+V_1^2+3V_2^2+3V_0V_1-3V_0V_2-5V_1V_2)V_{n+2}-(V_0^2+V_1^2+V_2^2+V_0V_1-3V_0V_2-V_1V_2)V_{n+1}$.

(h) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)l_n = -(V_0^2-3V_1^2-V_2^2-V_0V_1+V_0V_2+3V_1V_2)V_{n+2}-(V_0^2+V_1^2+V_2^2+V_0V_1-3V_0V_2-V_1V_2)V_{n+1}+(V_0^2-V_1^2+V_2^2+V_0V_1-V_0V_2-V_1V_2)V_n$.

(i) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)l_n = -(3V_0^2-5V_1^2-V_2^2-V_0V_1-V_0V_2+5V_1V_2)V_{n+1}+(V_0^2-V_1^2+V_2^2+V_0V_1-V_0V_2-V_1V_2)V_n+(V_0^2-3V_1^2-V_2^2-V_0V_1+V_0V_2+3V_1V_2)V_{n-1}$.

(j) $(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)l_n = -(5V_0^2-9V_1^2-3V_2^2-3V_0V_1-V_0V_2+11V_1V_2)V_n+(V_0^2-3V_1^2-V_2^2-V_0V_1+V_0V_2+3V_1V_2)V_{n-1}+(3V_0^2-5V_1^2-V_2^2-V_0V_1-V_0V_2+5V_1V_2)V_{n-2}$.

Next, we present a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 14. *The following equalities are true*

$$\begin{aligned}
5G_n &= 6H_{n+4} - 4H_{n+3} - 7H_{n+2}, \\
5G_n &= 8H_{n+3} - 7H_{n+2} - 6H_{n+1}, \\
5G_n &= 9H_{n+2} - 6H_{n+1} - 8H_n, \\
5G_n &= 12H_{n+1} - 8H_n - 9H_{n-1}, \\
5G_n &= 16H_n - 9H_{n-1} - 12H_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
H_n &= -3G_{n+4} + 10G_{n+3} - 8G_{n+2}, \\
H_n &= 4G_{n+3} - 8G_{n+2} + 3G_{n+1}, \\
H_n &= 3G_{n+1} - 4G_n, \\
H_n &= 2G_n - 3G_{n-2}.
\end{aligned}$$

Next, we give a few basic relations between $\{G_n\}$ and $\{l_n\}$.

Lemma 15. *The following equalities are true*

$$\begin{aligned}
2G_n &= -2l_{n+4} + 3l_{n+3} + l_{n+2}, \\
2G_n &= -l_{n+3} + l_{n+2} + 2l_{n+1}, \\
2G_n &= -l_{n+2} + 2l_{n+1} + l_n, \\
2G_n &= l_n + l_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
l_n &= -3G_{n+4} + 7G_{n+3} - 3G_{n+2}, \\
l_n &= G_{n+3} - 3G_{n+2} + 3G_{n+1}, \\
l_n &= -G_{n+2} + 3G_{n+1} - G_n, \\
l_n &= G_{n+1} - G_n + G_{n-1}, \\
l_n &= G_n + G_{n-1} - G_{n-2}.
\end{aligned}$$

Now, we present a few basic relations between $\{H_n\}$ and $\{l_n\}$.

Lemma 16. *The following equalities are true*

$$\begin{aligned}
2H_n &= 5l_{n+4} - 9l_{n+3} + 2l_{n+2}, \\
2H_n &= l_{n+3} + 2l_{n+2} - 5l_{n+1}, \\
2H_n &= 4l_{n+2} - 5l_{n+1} - l_n, \\
2H_n &= 3l_{n+1} - l_n - 4l_{n-1}, \\
2H_n &= 5l_n - 4l_{n-1} - 3l_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
5l_n &= 9H_{n+4} - 11H_{n+3} - 3H_{n+2}, \\
5l_n &= 7H_{n+3} - 3H_{n+2} - 9H_{n+1}, \\
5l_n &= 11H_{n+2} - 9H_{n+1} - 7H_n, \\
5l_n &= 13H_{n+1} - 7H_n - 11H_{n-1}, \\
5l_n &= 19H_n - 11H_{n-1} - 13H_{n-2}.
\end{aligned}$$

5 Relations Betwen Special Numbers

In this section we present identities on modified Leonardo, Leonardo-Lucas and Leonardo numbers and Fibonacci and Lucas numbers. We know that

$$\begin{aligned}
G_n &= F_{n+2} - 1 \\
H_n &= L_n + 1 \\
l_n &= 2F_{n+1} - 1
\end{aligned}$$

We also note that from Lemma 14, Lemma 14 and Lemma 16, we have, respectively,

$$\begin{aligned}
5G_n &= 9H_{n+2} - 6H_{n+1} - 8H_n, \\
H_n &= 3G_{n+1} - 4G_n, \\
5l_n &= 11H_{n+2} - 9H_{n+1} - 7H_n.
\end{aligned}$$

Using the above identities we see that

$$5G_n = 3L_{n+1} + L_n - 5, \tag{5.1}$$

$$H_n = 2F_{n+1} - F_n + 1, \tag{5.2}$$

$$5l_n = 2L_{n+1} + 4L_n - 5. \tag{5.3}$$

Using the above identities we obtain the following Binet's formula of generalized Leonardo numbers in the following forms.

$$\begin{aligned}
V_n &= (V_1-V_0)F_{n+2}+(V_2-2V_1+V_0)F_{n+1}-(V_2-V_1-V_0)\\
&= \frac{1}{5}(V_2+V_1-2V_0)L_{n+1}+\frac{1}{5}(2V_2-3V_1+V_0)L_n-(V_2-V_1-V_0)\\
&= (V_1-V_0)G_n+(V_2-2V_1+V_0)G_{n-1}+V_0\\
&= \frac{1}{5}(V_2+V_1-2V_0)H_{n+1}+\frac{1}{5}(2V_2-3V_1+V_0)H_n+\frac{1}{5}(-8V_2+7V_1+6V_0)\\
&= \frac{1}{2}(V_1-V_0)l_{n+1}+\frac{1}{2}(V_2-2V_1+V_0)l_n+\frac{1}{2}(-V_2+V_1+2V_0).
\end{aligned}$$

By Lemma 11, we know that

$$\begin{aligned}
&(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n\\
=\;& -(V_0^2-2V_1^2+V_1V_2)V_{n+2}+(V_2^2+V_0V_1-2V_1V_2)V_{n+1}+(-V_1^2+V_0V_2)V_n
\end{aligned}$$

so the identity

$$V_n=(V_1-V_0)G_n+(V_2-2V_1+V_0)G_{n-1}+V_0$$

can be written in the following form

$$\begin{aligned}
&(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)V_n\\
=\;& (V_1-V_0)(-(V_0^2-2V_1^2+V_1V_2)V_{n+2}+(V_2^2+V_0V_1-2V_1V_2)V_{n+1}+(-V_1^2+V_0V_2)V_n)\\
&+(V_2-2V_1+V_0)(-(V_0^2-2V_1^2+V_1V_2)V_{n+1}+(V_2^2+V_0V_1-2V_1V_2)V_n+(-V_1^2+V_0V_2)V_{n-1})\\
&+(V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)V_0
\end{aligned}$$

6 Special Identities

We now present a few special identities for the generalized Leonardo sequence $\{V_n\}$.

Theorem 17. *(Catalan's identity of the generalized Leonardo sequence) For all integers n and m, the following identity holds:*

$V_{n+m}V_{n-m}-V_n^2=((V_1-V_0)F_{n+m+2}+(V_2-2V_1+V_0)F_{n+m+1}-(V_2-V_1-V_0))((V_1-V_0)F_{n-m+2}+(V_2-2V_1+V_0)F_{n-m+1}-(V_2-V_1-V_0))-((V_1-V_0)F_{n+2}+(V_2-2V_1+V_0)F_{n+1}-(V_2-V_1-V_0))^2.$

Proof. We use the identity (Binet's formula of V_n)

$$V_n=(V_1-V_0)F_{n+2}+(V_2-2V_1+V_0)F_{n+1}-(V_2-V_1-V_0).\quad\square \tag{6.1}$$

As special cases of the above theorem, we have the following corollary.

Corollary 18. *For all integers n and m, the following identities hold:*

(a) $G_{n+m}G_{n-m} - G_n^2 = (F_{-m+n+2} - 1)F_{m+n+2} - F_{n-m+2} - F_{n+2}^2 + 2F_{n+2}$.

(b) $H_{n+m}H_{n-m} - H_n^2 = (F_{-m+n+2} - 3F_{-m+n+1} - 1)F_{m+n+2} + 3(-F_{-m+n+2} + 3F_{-m+n+1} + 1)F_{m+n+1} + 3F_{n-m+1} - F_{n-m+2} - F_{n+2}^2 - 9F_{n+1}^2 + 6F_{n+1}F_{n+2} + 2F_{n+2} - 6F_{n+1}$.

(c) $l_{n+m}l_{n-m} - l_n^2 = 2(2F_{-m+n+1} - 1)F_{m+n+1} - 2F_{n-m+1} - 4F_{n+1}^2 + 4F_{n+1}$.

Note that for $m = 1$ in Catalan's identity of the generalized Leonardo sequence, we get the Cassini identity for the generalized Leonardo sequnce.

Theorem 19. *(Cassini's identity of the generalized Leonardo sequence) For all integers n, the following identity holds:*

$V_{n+1}V_{n-1} - V_n^2 = (-V_2^2 - V_1^2 + V_0^2 + 3V_1V_2 - V_0V_2 - V_0V_1)F_{n+1}^2 + (V_2^2 + V_1^2 - V_0^2 - 3V_1V_2 + V_0V_2 + V_0V_1)F_n^2 + (V_2 - 3V_1 + 2V_0)(V_2 - V_1 - V_0)F_{n+1} + (2V_2 - 5V_1 + 3V_0)(-V_2 + V_1 + V_0)F_n + (V_2^2 + V_1^2 - V_0^2 - 3V_1V_2 + V_0V_2 + V_0V_1)F_{n+1}F_n$.

Corollary 20. *For all integers n, the following identities hold:*

(a) $G_{n+1}G_{n-1} - G_n^2 = F_{n+1}^2 - F_n^2 - F_nF_{n+1} - F_{n+1} + F_n$.

(b) $H_{n+1}H_{n-1} - H_n^2 = -5F_{n+1}^2 + 5F_n^2 + 5F_nF_{n+1} - 4F_{n+1} + 7F_n$.

(c) $l_{n+1}l_{n-1} - l_n^2 = 2(-2F_{n+1}^2 + 2F_n^2 + 2F_nF_{n+1} + F_{n+1} - 2F_n)$.

The d'Ocagne's and Melham's identities can also be obtained by using (6.1). The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of generalized Leonardo sequence $\{V_n\}$.

Theorem 21. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$V_{m+1}V_n - V_mV_{n+1} = (V_0 - V_1)(V_0 + V_1 - V_2)(F_{n+1} - F_{m+1}) + (V_0 - 2V_1 + V_2)(V_0 + V_1 - V_2)(F_m - F_n) + (V_0^2 - V_1^2 - V_2^2 - V_0V_1 - V_0V_2 + 3V_1V_2)(F_nF_{m+1} - F_mF_{n+1})$.

(b) *(Melham's identity)*

$V_{n+1}V_{n+2}V_{n+6} - V_{n+3}^3 = (V_0^2 - V_1^2 - V_2^2 - V_0V_1 - V_0V_2 + 3V_1V_2)(-(V_1 - V_2)F_{n+1}^3 + (V_0 - V_1)F_n^3 + (V_0 - V_2)F_{n+1}F_n^2 - (V_0 - 2V_1 + V_2)F_{n+1}^2F_n) + (V_0 + V_1 - V_2)((5V_0 + 2V_1 - 7V_2)(V_0 + V_1 - 2V_2)F_{n+1}^2 + (V_0 - V_2)(2V_0 + 3V_1 - 5V_2)F_n^2 + (7V_0^2 + 3V_1^2 + 17V_2^2 + 7V_0V_1 - 21V_0V_2 - 13V_1V_2)F_{n+1}F_n) + (V_0 + V_1 - V_2)^2(-(4V_0 + 3V_1 - 7V_2)F_{n+1} - (3V_0 + V_1 - 4V_2)F_n)$.

Proof. Use the identity (6.1). □

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's and Melham's identities of modified Leonardo sequence $\{G_n\}$.

Corollary 22. *Let* n *and* m *be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$G_{m+1}G_n - G_mG_{n+1} = (F_{n+1} - F_{m+1}) + (F_nF_{m+1} - F_mF_{n+1}).$$

(b) *(Melham's identity)*

$$G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = F_{n+1}^3 - F_n^3 - 36F_{n+1}^2 - 14F_n^2 - 2F_n^2F_{n+1} - 45F_nF_{n+1} + 11F_{n+1} + 7F_n.$$

Second one presents d'Ocagne's and Melham's identities of Leonardo-Lucas sequence $\{H_n\}$.

Corollary 23. *Let* n *and* m *be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_mH_{n+1} = (F_{n+1} - F_{m+1}) + 3(F_m - F_n) - 5(F_nF_{m+1} - F_mF_{n+1}).$$

(b) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = -10F_{n+1}^3 - 5F_n^3 + 27F_{n+1}^2 + 8F_n^2 + 15F_nF_{n+1}^2 + 5F_n^2F_{n+1} + 33F_nF_{n+1} + 10F_{n+1} + 5F_n.$$

Third one presents d'Ocagne's and Melham's identities of Leonardo sequence $\{l_n\}$.

Corollary 24. *Let* n *and* m *be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$l_{m+1}l_n - l_ml_{n+1} = -2(F_m - F_n) - 4(F_nF_{m+1} - F_mF_{n+1}).$$

(b) *(Melham's identity)*

$$l_{n+1}l_{n+2}l_{n+6} - l_{n+3}^3 = 2(-4F_{n+1}^3 - 28F_{n+1}^2 - 10F_n^2 + 4F_nF_{n+1}^2 + 4F_n^2F_{n+1} + 7F_{n+1} + 4F_n - 34F_nF_{n+1}).$$

7 On the Recurrence Properties of Generalized Leonardo Sequence

Taking $r = 2, s = 0, t = -1$ **in Theorem 5 (a) and (b), we obtain the following Proposition.**

Proposition 25. *For $n \in \mathbb{Z}$, generalized Leonardo numbers (the case $r = 2, s = 0, t = -1$) have the following identity:*

$$V_{-n} = (-1)^{-n}(V_{2n} - H_n V_n + \frac{1}{2}(H_n^2 - H_{2n})V_0)$$

From the above Proposition and Corollary 6, we have the following corollary which gives the connection between the special cases of generalized Leonardo sequence at the positive index and the negative index: for modified Leonardo, Leonardo-Lucas and Leonardo numbers: take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2,$ take $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 4,$ and $W_n = l_n$ with $l_0 = 1, l_1 = 1, l_2 = 3,$ respectively. Note that in this case $H_n = H_n$.

Corollary 26. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *modified Leonardo sequence:*

$$G_{-n} = \frac{1}{(-1)^n}(4G_n^2 + G_{2n} - 3G_{n+1}G_n).$$

(b) *Leonardo-Lucas sequence:*

$$H_{-n} = \frac{1}{2(-1)^n}(H_n^2 - H_{2n}).$$

(c) *Leonardo sequence:*

$$l_{-n} = (-1)^{-n}(l_{2n} - H_n l_n + \frac{1}{2}(H_n^2 - H_{2n})).$$

By using the identity $2H_n = 4l_{n+2} - 5l_{n+1} - l_n$ (and Proposition 25), we get

$$l_{-n} = \frac{1}{8}(-1)^{-n}(16l_{n+2}^2 + 25l_{n+1}^2 + 5l_n^2 - 40l_{n+1}l_{n+2} - 24l_n l_{n+2} + 30l_n l_{n+1} - 8l_{2n+2} + 10l_{2n+1} + 10l_{2n}).$$

Note also that since $l_n = 2F_{n+1} - 1$ and $F_{-n} = (-1)^{n+1}F_n$, we get

$$\begin{aligned} l_{-n} &= 2F_{-n+1} - 1 = 2(-1)^n F_{n-1} - 1 \\ &= (-1)^n(l_{n-2} + 1) - 1. \end{aligned}$$

8 Sums

The following Corollary gives sum formulas of Fibonacci and Lucas numbers.

Corollary 27. *For $n \geq 0$, Fibonacci and Lucas numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} F_k = 2F_n + F_{n-1} - 1$.

(b) $\sum_{k=0}^{n} F_{2k} = 2F_{2n} - F_{2n-2} - 1$.

(c) $\sum_{k=0}^{n} F_{2k+1} = 2F_{2n+1} - F_{2n-1}$.

2.

(a) $\sum_{k=0}^{n} L_k = 2L_n + L_{n-1} - 1$.

(b) $\sum_{k=0}^{n} L_{2k} = 2L_{2n} - L_{2n-2} + 1$.

(c) $\sum_{k=0}^{n} L_{2k+1} = 2L_{2n+1} - L_{2n-1} - 2$.

Proof. It is given in Soykan [20, Corollary 4.5.]. $\square$

The following Corollary presents sum formulas of modified Leonardo, Leonardo-Lucas and Leonardo numbers.

Corollary 28. *For $n \geq 0$, modified Leonardo, Leonardo-Lucas and Leonardo numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} G_k = 3F_{n+1} + 2F_n - 3 - n$.

(b) $\sum_{k=0}^{n} G_{2k} = 2F_{2n+1} + F_{2n} - 2 - n$.

(c) $\sum_{k=0}^{n} G_{2k+1} = 3F_{2n+1} + 2F_{2n} - 2 - n$.

2.

(a) $\sum_{k=0}^{n} H_k = 2L_n + L_{n-1} + n$.

(b) $\sum_{k=0}^{n} H_{2k} = 2L_{2n} - L_{2n-2} + 2 + n$.

(c) $\sum_{k=0}^{n} H_{2k+1} = 2L_{2n+1} - L_{2n-1} - 1 + n$.

3.

(a) $\sum_{k=0}^{n} l_k = 4F_{n+1} + 2F_n - 3 - n$.

(b) $\sum_{k=0}^{n} l_{2k} = 2(F_{2n+1} + F_{2n}) - (n+1)$.

(c) $\sum_{k=0}^{n} l_{2k+1} = 4F_{2n+1} + 2F_{2n} - 3 - n$.

Proof. The proof follows from Corollary 27 and the identities (2.9), (2.10) and (2.11), i.e.,

$$\begin{aligned} G_n &= F_{n+2} - 1, \\ H_n &= L_n + 1, \\ l_n &= 2F_{n+1} - 1. \quad \square \end{aligned}$$

The following Corollary gives sum formulas of squares of Fibonacci and Lucas numbers.

Corollary 29. *For $n \geq 0$, Fibonacci numbers and Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} F_k^2 = 2F_n^2 - F_{n-1}^2 + (-1)^n$.

(b) $\sum_{k=0}^{n} L_k^2 = 2L_n^2 - L_{n-1}^2 - 5(-1)^n + 2$.

Proof. It is given in Soykan [21, Corollary 2.3.]. $\square$

The following Corollary presents sum formulas of squares of modified Leonardo, Leonardo-Lucas and Leonardo numbers.

Corollary 30. *For $n \geq 0$, modified Leonardo, Leonardo-Lucas and Leonardo numbers have the following properties:*

(a) $\sum_{k=0}^{n} G_k^2 = F_{n+1}^2 + 2F_n^2 + 4F_nF_{n+1} - 6F_{n+1} - 4F_n + (-1)^n + 4 + n$.

(b) $\sum_{k=0}^{n} H_k^2 = 2L_n^2 - L_{n-1}^2 + 4L_n + 2L_{n-1} - 5(-1)^n + 1 + n$.

(b) $\sum_{k=0}^{n} l_k^2 = 4F_n^2 + 8F_nF_{n+1} - 8F_{n+1} - 4F_n + 4(-1)^n + 5 + n$.

Proof. The proof follows from Corollary 29, Corollary 27 and the identities (2.9), (2.10) and (2.11). $\square$

The following Corollary gives sum formulas of cubes of Fibonacci and Lucas numbers.

Corollary 31. *For $n \geq 0$, Fibonacci numbers and Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} F_k^3 = \frac{1}{20}(25F_n^3 + 5F_{n-1}^3 + 3(-1)^n(9F_n - F_{n-1} - 4F_{n+1}) + 10)$.

(b) $\sum_{k=0}^{n} L_k^3 = \frac{1}{4}(5L_n^3 + L_{n-1}^3 + 3(-1)^n(4L_{n+1} - 9L_n + L_{n-1}) + 38)$.

Proof. It is given in Soykan [22, Corollary 2.3.]

The following Corollary presents sum formulas of cubes of modified Leonardo, Leonardo-Lucas and Leonardo numbers.

Corollary 32. *For $n \geq 0$, modified Leonardo, Leonardo-Lucas and Leonardo numbers have the following properties:*

(a) $\sum_{k=0}^{n} G_k^3 = \frac{1}{4}(9F_{n+1}^3 - 12F_{n+1}^2 + 8F_n^3 - 24F_n^2 + 9F_nF_{n+1}^2 + 15F_n^2F_{n+1} - 48F_nF_{n+1} + 36F_{n+1} + 24F_n + 3(-1)^n(-F_{n+1} + 2F_n - 4) - 18 - 4n)$.

(b) $\sum_{k=0}^{n} H_k^3 = \frac{1}{4}(5L_n^3 + 24L_n^2 + L_{n-1}^3 - 12L_{n-1}^2 + 24L_n + 12L_{n-1} + 3(-1)^n(4L_{n+1} - 9L_n + L_{n-1} - 20) + 54 + 4n)$.

(c) $\sum_{k=0}^{n} l_k^3 = 10F_{n+1}^3 + 8F_n^3 - 12F_n^2 - 6F_nF_{n+1}^2 + 6F_n^2F_{n+1} - 24F_nF_{n+1} + 12F_{n+1} + 6F_n + 6(-1)^n(2F_n - F_{n+1} - 2) - 3 - n$

Proof. The proof follows from Corollary 31, Corollary 29, Corollary 27 and the identities (2.9), (2.10) and (2.11). $\square$

9 Matrices Related With Generalized Leonardo numbers

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -1$. **From (2.1) we have**

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{9.1}$$

and from (1.6) (or using (9.1) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = G$ **in (9.1) we have**

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{9.2}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & -V_{n-1} & -V_n \\ V_n & -V_{n-2} & -V_{n-1} \\ V_{n-1} & -V_{n-3} & -V_{n-2} \end{pmatrix}$$

Theorem 33. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 2, s = 0, t = -1$ in Soykan [18, Theorem 5.1.]. $\square$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} - A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 1$$

for all integer m and n.

Corollary 34. *For all integers n, we have the following formulas for the modified Leonardo, Leonardo-Lucas and Leonardo numbers.*

(a) *Modified Leonardo Numbers.*

$$A^n = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

(b) *Leonardo-Lucas Numbers.*

$$A^n = \frac{1}{5}\begin{pmatrix} 9H_{n+3} - 6H_{n+2} - 8H_{n+1} & -(9H_{n+1} - 6H_n - 8H_{n-1}) & -(9H_{n+2} - 6H_{n+1} - 8H_n) \\ 9H_{n+2} - 6H_{n+1} - 8H_n & -(9H_n - 6H_{n-1} - 8H_{n-2}) & -(9H_{n+1} - 6H_n - 8H_{n-1}) \\ (9H_{n+1} - 6H_n - 8H_{n-1}) & -(9H_{n-1} - 6H_{n-2} - 8H_{n-3}) & -(9H_n - 6H_{n-1} - 8H_{n-2}) \end{pmatrix}.$$

(c) *Leonardo Numbers.*

$$A^n = \frac{1}{2}\begin{pmatrix} -l_{n+3} + 2l_{n+2} + l_{n+1} & -(-l_{n+1} + 2l_n + l_{n-1}) & -(-l_{n+2} + 2l_{n+1} + l_n) \\ -l_{n+2} + 2l_{n+1} + l_n & -(-l_n + 2l_{n-1} + l_{n-2}) & -(-l_{n+1} + 2l_n + l_{n-1}) \\ -l_{n+1} + 2l_n + l_{n-1} & -(-l_{n-1} + 2l_{n-2} + l_{n-3}) & -(-l_n + 2l_{n-1} + l_{n-2}) \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 33 (a).

(b) Note that, from Lemma 14, we have

$$5G_n = 9H_{n+2} - 6H_{n+1} - 8H_n.$$

Using the last equation and (a), we get required result.

(c) Note that, from Lemma 15, we have

$$2G_n = -l_{n+2} + 2l_{n+1} + l_n.$$

Using the last equation and (a), we get required result. $\square$

Using the above last Corollary and the identities (2.9), (2.10) and (2.11) we obtain the following identities for Fibonacci and Lucas numbers.

Corollary 35. *For all integers n, we have the following formulas for Fibonacci and Lucas numbers.*

(a) *Fibonacci Numbers.*

$$A^n = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+3}-1 & -F_{n+1}+1 & -F_{n+2}+1 \\ F_{n+2}-1 & -F_n+1 & -F_{n+1}+1 \\ F_{n+1}-1 & -F_{n-1}+1 & -F_n+1 \end{pmatrix}.$$

(b) *Lucas Numbers.*

$$A^n = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{5}\begin{pmatrix} 3L_{n+2}+L_{n+1}-5 & -3L_n-L_{n-1}+5 & -3L_{n+1}-L_n+5 \\ 3L_{n+1}+L_n-5 & -3L_{n-1}-L_{n-2}+5 & -3L_n-L_{n-1}+5 \\ 3L_n+L_{n-1}-5 & -3L_{n-2}-L_{n-3}+5 & -3L_{n-1}-L_{n-2}+5 \end{pmatrix}.$$

Theorem 36. *For all integers m, n, we have*

$$V_{n+m} = V_nG_{m+1} - V_{n-1}G_{m-1} - V_{n-2}G_m \tag{9.3}$$

Proof. Take $r = 2, s = 0, t = -1$ in Soykan [18, Theorem 5.2.]. $\square$

By Lemma 11, we know that

$$\begin{aligned} & (V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)G_n \\ = \; & -(V_0^2-2V_1^2+V_1V_2)V_{n+2}+(V_2^2+V_0V_1-2V_1V_2)V_{n+1}+(-V_1^2+V_0V_2)V_n, \end{aligned}$$

so (9.3) can be written in the following form

$$\begin{aligned} & (V_0+V_1-V_2)(V_0^2-V_1^2-V_2^2-V_0V_1-V_0V_2+3V_1V_2)V_{n+m} \\ = \; & V_n(-(V_0^2-2V_1^2+V_1V_2)V_{m+3}+(V_2^2+V_0V_1-2V_1V_2)V_{m+2}+(-V_1^2+V_0V_2)V_{m+1}) \\ & -V_{n-1}(-(V_0^2-2V_1^2+V_1V_2)V_{m+1}+(V_2^2+V_0V_1-2V_1V_2)V_m+(-V_1^2+V_0V_2)V_{m-1}) \\ & -V_{n-2}(-(V_0^2-2V_1^2+V_1V_2)V_{m+2}+(V_2^2+V_0V_1-2V_1V_2)V_{m+1}+(-V_1^2+V_0V_2)V_m). \end{aligned}$$

Corollary 37. *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_nG_{m+1}-G_{n-1}G_{m-1}-G_{n-2}G_m, \\ H_{n+m} &= H_nG_{m+1}-H_{n-1}G_{m-1}-H_{n-2}G_m, \\ l_{n+m} &= l_nG_{m+1}-l_{n-1}G_{m-1}-l_{n-2}G_m, \end{aligned}$$

and

$$5H_{m+n} = -H_n(-9H_{m+3}+6H_{m+2}+8H_{m+1})+H_{n-1}(-9H_{m+1}+6H_m+8H_{m-1})+H_{n-2}(-9H_{m+2}+6H_{m+1}+8H_m),$$

and

$$2l_{m+n} = l_n(-l_{m+3} + 2l_{m+2} + l_{m+1}) - l_{n-1}(-l_{m+1} + 2l_m + l_{m-1}) - l_{n-2}(-l_{m+2} + 2l_{m+1} + l_m).$$

Taking $m = n$ in the last corollary we obtain the following identities:

$$\begin{aligned}
G_{2n} &= G_n (G_{n+1} - G_{n-2}) - G_{n-1}^2, \\
H_{2n} &= H_n G_{n+1} - H_{n-1} G_{n-1} - H_{n-2} G_n, \\
l_{2n} &= l_n G_{n+1} - l_{n-1} G_{n-1} - l_{n-2} G_n, \\
5H_{2n} &= H_n(9H_{n+3} - 6H_{n+2} - 8H_{n+1} + 6H_{n-1} + 8H_{n-2}) \\
&\quad + H_{n-1}(8H_{n-1} - 9H_{n+1}) + 3H_{n-2}(2H_{n+1} - 3H_{n+2}), \\
2l_{2n} &= -l_{n-1}^2 + l_n(-l_{n+3} + 2l_{n+2} + l_{n+1} - 2l_{n-1} - l_{n-2}) + l_{n+1}(l_{n-1} - 2l_{n-2}) + l_{n+2} l_{n-2}.
\end{aligned}$$

Taking $m = n + 1$ in the last corollary we obtain the following identities:

$$\begin{aligned}
G_{2n+1} &= G_n G_{n+2} - G_{n-1} G_n - G_{n-2} G_{n+1}, \\
H_{2n+1} &= H_n G_{n+2} - H_{n-1} G_n - H_{n-2} G_{n+1}, \\
l_{2n+1} &= l_n G_{n+2} - l_{n-1} G_n - l_{n-2} G_{n+1}, \\
5H_{2n+1} &= H_n(9H_{n+4} - 6H_{n+3} - 8H_{n+2} + 8H_{n-1}) \\
&\quad + 2H_{n+1}(4H_{n-2} + 3H_{n-1}) + 3H_{n-2}(-3H_{n+3} + 2H_{n+2}) - 9H_{n-1} H_{n+2}, \\
2l_{2n+1} &= -l_n(l_{n+4} - 2l_{n+3} - l_{n+2} + l_{n-1}) - l_{n+1}(l_{n-2} + 2l_{n-1}) + l_{n+2}(l_{n-1} - 2l_{n-2}) + l_{n+3} l_{n-2}.
\end{aligned}$$

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