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Generalized Pierre Numbers

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Abstract

In this paper, we introduce and investigate the generalized Pierre sequences and we deal with, in detail, two special cases, namely, Pierre and Pierre-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Pierre, Pierre-Lucas and Tribonacci, Tribonacci-Lucas numbers.

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Keywords

Pierre numbers, Pierre-Lucas numbers, Tribonacci numbers, Tribonacci-Lucas numbers.

1. Introduction

Tribonacci sequence $\{T_n\}_{n\geq 0}$ (OEIS: A000073, [15]) and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$

(OEIS: A001644, [15]) are defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{1.1}$$

and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \tag{1.2}$$

respectively. These sequences has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences.

Basic properties of these sequences are given in [1,2,3,4,5,8,11,12,13,16,19,25,26,27].

The sequences $\{T_n\}_{n\geq 0}$, and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} T_{-n} &= -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \\ K_{-n} &= -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1)-(1.2) hold for all integer n.

Now, we define two sequences related to Tribonacci and Tribonacci-Lucas numbers. Pierre and Pierre-Lucas numbers are defined as

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + 1, \qquad \text{with} \quad P_0 = 0, P_1 = 1, P_2 = 2 \quad n \geq 3,$$

and

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} - 2, \qquad \text{with} \quad C_0 = 4, C_1 = 2, C_2 = 4 \quad n \geq 3,$$

respectively. The first few values of Pierre and Pierre-Lucas numbers are

$$0, 1, 2, 4, 8, 15, 28, 52, 96, 177, 326...$$

and

$$4, 2, 4, 8, 12, 22, 40, 72, 132, 242, 444, ...$$

respectively. The sequences $\{P_n\}$ and $\{C_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned} P_n &= 2P_{n-1} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, \ P_3 = 4, \qquad n \geq 4, \\ C_n &= 2C_{n-1} - C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \qquad n \geq 4. \end{aligned}$$

There are close relations between Pierre, Pierre-Lucas and Tribonacci, Tribonacci-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} 2P_n &= T_{n+2} + T_n - 1, \\ 22P_n &= 4K_{n+2} + 2K_{n+1} - K_n - 11, \\ C_n &= -T_{n+2} + 4T_{n+1} - T_n + 1, \\ C_n &= K_n + 1, \end{aligned}$$

and

$$\begin{aligned} T_{n+1} &= P_{n+1}-P_n, \\ 22T_n &= 5C_{n+2}-3C_{n+1}-4C_n+2, \\ K_n &= 4P_{n+1}-6P_n-1, \\ 2K_n &= C_{n+3}-C_{n+2}-C_{n+1}+C_n. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Pierre, Pierre-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r,s,t,u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3;r,s,t,u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}, \qquad W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3, \quad n\geq 4 \tag{1.3}$$

where W_0,W_1,W_2,W_3 are arbitrary complex (or real) numbers and r,s,t,u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [6,9,10,14,20,22,23,28,29]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n=1,2,3,...$ when $u\neq 0$. Therefore, recurrence (1.3) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4-rz^3-sz^2-tz-u=0 \tag{1.4}$$

whose roots are $\alpha,\beta,\gamma,\delta$. Note that we have the following identities

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= r, \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -s, \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1. *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$) Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.5}$$

where

$$
\begin{aligned}
p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0,\\
p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0,\\
p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0,\\
p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0.
\end{aligned}
$$

Usually, it is customary to choose $\alpha,\beta,\gamma,\delta$ so that the Equ. (1.4) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.4) for non-negative integers is valid for all integers n (see [7]).

Next, we consider two special cases of the generalized (r,s,t,u) sequence $\{W_n\}$ which we call them (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas sequences. (r,s,t,u)-Fibonacci sequence $\{G_n\}_{n\geq0}$ and (r,s,t,u)-Lucas sequence $\{H_n\}_{n\geq0}$ are defined, respectively, by the fourth-order recurrence relations

$$
\begin{aligned}
G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, &(1.6)\\
G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s,\\
H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, &(1.7)\\
H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t.
\end{aligned}
$$

The sequences $\{G_n\}_{n\geq0}$ and $\{H_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)},\\
H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)},
\end{aligned}
$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.6) and (1.7) hold for all integers n.

For all integers n, (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers (using initial conditions in (1.6) or (1.7)) can be expressed using Binet's formulas as in the following corollary.

Corollary 2. *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$) Binet's formula of (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers are*

$$
G_n=\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
$$

and

$$
H_n=\alpha^n+\beta^n+\gamma^n+\delta^n,
$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1, respectively. □

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 3. *Suppose that $f_{W_n}(z) = \sum\limits_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized (r,s,t,u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \tag{1.8}$$

Proof. For a proof, see Soykan [20, Lemma 1]. □

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 4 (Simson's Formula of Generalized (r,s,t,u) Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{1.9}$$

Proof. **(1.9) is given in Soykan [18].** □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 5. *For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r,s,t,u)-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_n W_{2n} - 3H_n^2 W_n + 3H_{2n}W_n + W_0 H_n^3 + 2W_0 H_{3n} - 3W_0 H_n H_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_n W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [21, Theorem 1.]. □

Using Theorem 5, we have the following corollary, see Soykan [21, Corollary 4].

Corollary 6. *For $n \in \mathbb{Z}$, we have*

(a) $2(-u)^{n+4} G_{-n} = -(3ru^2 + t^3 - 3stu)^2 G_n^3 - (2su - t^2)^2 G_{n+3}^2 G_n - (-rt^2 - tu + 2rsu)^2 G_{n+2}^2 G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2 G_{n+1}^2 G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n$.

(b) $H_{-n}=\frac{1}{6}\left(-u\right)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0=0$ and $H_0=4$ in Theorem 5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n}+6H_nG_{2n}-3H_n^2G_n+3H_{2n}G_n), \tag{1.10}$$

$$H_{-n} = \frac{1}{6}\left(-u\right)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right), \tag{1.11}$$

respectively.

If we define the square matrix A of order 4 as

$$A=A_{rstu}=\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n=\begin{pmatrix} G_{n+1} & sG_n+tG_{n-1}+uG_{n-2} & tG_n+uG_{n-1} & uG_n \\ G_n & sG_{n-1}+tG_{n-2}+uG_{n-3} & tG_{n-1}+uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2}+tG_{n-3}+uG_{n-4} & tG_{n-2}+uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3}+tG_{n-4}+uG_{n-5} & tG_{n-3}+uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n=\begin{pmatrix} W_{n+1} & sW_n+tW_{n-1}+uW_{n-2} & tW_n+uW_{n-1} & uW_n \\ W_n & sW_{n-1}+tW_{n-2}+uW_{n-3} & tW_{n-1}+uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2}+tW_{n-3}+uW_{n-4} & tW_{n-2}+uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3}+tW_{n-4}+uW_{n-5} & tW_{n-3}+uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 7. *For all integers m,n, we have*

(a) $B_n=A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n=\begin{pmatrix} G_{n+1} & sG_n+tG_{n-1}+uG_{n-2} & tG_n+uG_{n-1} & uG_n \\ G_n & sG_{n-1}+tG_{n-2}+uG_{n-3} & tG_{n-1}+uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2}+tG_{n-3}+uG_{n-4} & tG_{n-2}+uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3}+tG_{n-4}+uG_{n-5} & tG_{n-3}+uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b) $U_1A^n=A^nU_1.$

(c) $U_{n+m}=U_nB_m=B_mU_n.$

Proof. For the proof, see Soykan [20, Theorem 19]. □

Theorem 8. *For all integers $m, n,$ we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.12}$$

Proof. For the proof, see Soykan [20, Theorem 20]. □

In the next sections, we present new results.

2 Generalized Pierre Sequence

In this paper, we consider the case $r = 2, s = 0, t = 0, u = -1$. A generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 2, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 0, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned} \alpha+\beta+\gamma &= 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

The first few generalized Pierre numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pierre numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_2-W_3$
2	W_2	$2W_1-W_2$
3	W_3	$2W_0-W_1$
4	$2W_3-W_0$	$4W_2-W_0-2W_3$
5	$4W_3-W_1-2W_0$	$4W_1-4W_2+W_3$
6	$8W_3-2W_1-W_2-4W_0$	$4W_0-4W_1+W_2$
7	$15W_3-4W_1-2W_2-8W_0$	$W_1-4W_0+8W_2-4W_3$
8	$28W_3-8W_1-4W_2-15W_0$	$W_0+8W_1-12W_2+4W_3$
9	$52W_3-15W_1-8W_2-28W_0$	$8W_0-12W_1+6W_2-W_3$
10	$96W_3-28W_1-15W_2-52W_0$	$6W_1-12W_0+15W_2-8W_3$
11	$177W_3-52W_1-28W_2-96W_0$	$6W_0+15W_1-32W_2+12W_3$
12	$326W_3-96W_1-52W_2-177W_0$	$15W_0-32W_1+24W_2-6W_3$
13	$600W_3-177W_1-96W_2-326W_0$	$24W_1-32W_0+24W_2-15W_3$

Note that the sequences $\{P_n\}$ and $\{C_n\}$ which are defined in the section Introduction, are the special cases of the generalized Pierre sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$, in this section as well. Pierre sequence $\{P_n\}_{n\geq 0}$ and Pierre-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} P_n &= 2P_{n-1}-P_{n-4}, \qquad P_0=0, P_1=1, P_2=2,\ P_3=4, \qquad n\geq 4, \\ C_n &= 2C_{n-1}-C_{n-4}, \qquad C_0=4, C_1=2, C_2=4, C_3=8, \qquad n\geq 4. \end{aligned}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} P_{-n} &= 2P_{-(n-3)}-P_{-(n-4)}, \\ C_{-n} &= 2C_{-(n-3)}-C_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the Pierre and Pierre-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	4	8	15	28	52	96	177	326	600	1104	2031
P_{-n}	0	0	0	-1	0	0	-2	1	0	-4	4	-1	-8	12
C_n	4	2	4	8	12	22	40	72	132	242	444	816	1500	2758
C_{-n}	4	0	0	6	-4	0	12	-14	4	24	-40	22	44	-104

(1.5) can be used to obtain the Binet formula of generalized Pierre numbers. Binet's formula of generalized Pierre numbers can be given as follows:

Theorem 9. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of generalized Pierre numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)\alpha^n}{2\alpha^2+2\alpha-2} \\ &\quad + \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)\beta^n}{2\beta^2+2\beta-2} \\ &\quad + \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)\gamma^n}{2\gamma^2+2\gamma-2} \\ &\quad - \frac{W_3 - W_2 - W_1 - W_0}{2}. \end{aligned}$$

Pierre and Pierre-Lucas numbers can be expressed using Binet's formulas as follows.

Corollary 10. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of Pierre and Pierre-Lucas numbers are*

$$P_n = \frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

Note that Binet's formulas of Tribonacci and Tribonacci-Lucas numbers, respectively, are

$$\begin{aligned} T_n &= \frac{(\alpha-1)}{2(\alpha^3-2)}\alpha^{n+2} + \frac{(\beta-1)}{2(\beta^3-2)}\beta^{n+2} + \frac{(\gamma-1)}{2(\gamma^3-2)}\gamma^{n+2} \\ &= \frac{(\alpha-1)}{2(\alpha^2+\alpha-1)}\alpha^{n+2} + \frac{(\beta-1)}{2(\beta^2+\beta-1)}\beta^{n+2} + \frac{(\gamma-1)}{2(\gamma^2+\gamma-1)}\gamma^{n+2}, \\ K_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

see, Soykan [19] for more details.

So, by using Binet's formulas of Pierre, Pierre-Lucas and Tribonacci, Tribonacci-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

Lemma 11. *For all integers n, the following equalities (identities) are true:*

(a)

- $T_{n+3} = P_{n+3} - P_{n+2}$.
- $T_n = P_{n+3} - 2P_{n+2} + P_n$.
- $2P_{n+4} = 9T_{n+2} + 8T_{n+1} + 5T_n - 1$.
- $2P_n = T_{n+2} + T_n - 1$.
- $T_n = -P_{n+2} + P_{n+1} + 2P_n + 1$.
- $T_{n+1} = P_{n+1} - P_n$.

(b)

- $22T_{n+3} = 9C_{n+3} + C_{n+2} - 4C_{n+1} - 6C_n$.
- $22T_n = -C_{n+3} + 6C_{n+2} - 2C_{n+1} - 3C_n$.
- $C_{n+4} = 7T_{n+2} + 4T_{n+1} + 3T_n + 1$.
- $C_n = -T_{n+2} + 4T_{n+1} - T_n + 1$.
- $22T_n = 5C_{n+2} - 3C_{n+1} - 4C_n + 2$.
- $C_{n+1} + 3C_n = 10T_{n+1} - 4T_n + 4$.

(c)

- $K_{n+3} = P_{n+3} + P_{n+2} + P_{n+1} - 3P_n$.
- $K_n = -P_{n+3} + P_{n+2} + 5P_{n+1} - 5P_n$.
- $22P_{n+4} = 34K_{n+2} + 28K_{n+1} + 19K_n - 11$.
- $22P_n = 4K_{n+2} + 2K_{n+1} - K_n - 11$.
- $K_n = 4P_{n+1} - 6P_n - 1$.

(d)

- $2K_{n+3} = 3C_{n+3} - C_{n+2} - C_{n+1} - C_n$.

- $2K_n = C_{n+3} - C_{n+2} - C_{n+1} + C_n$.
- $C_{n+4} = 2K_{n+2} + 2K_{n+1} + K_n + 1$.
- $C_n = K_n + 1$.
- $K_n = C_n - 1$.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n z^n$ **of the sequence** W_n.

Lemma 12. *Suppose that* $f_{W_n}(z) = \sum\limits_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Pierre sequence* $\{W_n\}$. *Then,* $\sum\limits_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 2W_0)z + (W_2 - 2W_1)z^2 + (W_3 - 2W_2)z^3}{1 - 2z + z^4}.$$

Proof. Take $r = 2, s = 0, t = 0, u = -1$ **in Lemma 3.**

The previous lemma gives the following results as particular examples.

Corollary 13. *Generating functions of Pierre and Pierre-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} P_n z^n &= \frac{z}{1 - 2z + z^4} = \frac{z}{(-1 + z + z^2 + z^3)(z - 1)}, \\ \sum_{n=0}^{\infty} C_n z^n &= \frac{-6z + 4}{1 - 2z + z^4} = \frac{-6z + 4}{(-1 + z + z^2 + z^3)(z - 1)}, \end{aligned}$$

respectively.

3 Simson Formulas

Now, we present Simson's formula of generalized Pierre numbers.

Theorem 14 (Simson's Formula of Generalized Pierre Numbers)**.** *For all integers* n, *we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (W_3 - W_2 - W_1 - W_0)(W_3^3 - W_2^3 - W_1^3 - W_0^3 + (-5W_2 + W_1 + W_0)W_3^2 + (7W_3 - 3W_0 - W_1)W_2^2 + (3W_3 + W_2 - W_0)W_1^2 + (W_3 + W_2 + W_1)W_0^2 + 4(-W_2W_3 - W_0W_3 + W_0W_2)W_1).$$

Proof. Take $r = 2, s = 0, t = 0, u = -1$ **in Theorem 4.** $\square$

The previous theorem gives the following results as particular examples.

Corollary 15. *For all integers n, the Simson's formulas of Pierre and Pierre-Lucas numbers are given as*

$$\begin{vmatrix} P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} \end{vmatrix} = 1,$$

$$\begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_n \\ C_{n+2} & C_{n+1} & C_n & C_{n-1} \\ C_{n+1} & C_n & C_{n-1} & C_{n-2} \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} = -176,$$

respectively.

4 Some Identities

In this section, we obtain some identities of Pierre and Pierre-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{P_n\}$.

Lemma 16. *The following equalities are true:*

(a) $W_n = (4W_2 - W_0 - 2W_3)P_{n+5} + (4W_0 - W_1 - 8W_2 + 4W_3)P_{n+4} + (4W_1 - 4W_0 - W_2)P_{n+3} + (4W_2 - 4W_1 - W_3)P_{n+2}$.

(b) $W_n = (2W_0 - W_1)P_{n+4} + (4W_1 - 4W_0 - W_2)P_{n+3} + (4W_2 - 4W_1 - W_3)P_{n+2} + (W_0 - 4W_2 + 2W_3)P_{n+1}$.

(c) $W_n = (2W_1 - W_2)P_{n+3} + (4W_2 - 4W_1 - W_3)P_{n+2} + (W_0 - 4W_2 + 2W_3)P_{n+1} + (W_1 - 2W_0)P_n$.

(d) $W_n = (2W_2 - W_3)P_{n+2} + (W_0 - 4W_2 + 2W_3)P_{n+1} + (W_1 - 2W_0)P_n + (W_2 - 2W_1)P_{n-1}$.

(e) $W_n = W_0P_{n+1} + (W_1 - 2W_0)P_n + (W_2 - 2W_1)P_{n-1} + (W_3 - 2W_2)P_{n-2}$.

(f) $(W_0 + W_1 + W_2 - W_3)(W_0^3 + W_1^3 + W_2^3 - W_3^3 + W_0W_1^2 - W_0^2W_1 + 3W_0W_2^2 - W_0^2W_2 - W_0W_3^2 + W_1W_2^2 - W_0^2W_3 - W_1^2W_2 - W_1W_3^2 - 3W_1^2W_3 + 5W_2W_3^2 - 7W_2^2W_3 - 4W_0W_1W_2 + 4W_0W_1W_3 + 4W_1W_2W_3)P_n = -(W_0^3 + 2W_1^3 + 4W_2^3 + W_0W_2^2 - W_1^2W_2 + W_2W_3^2 - 4W_2^2W_3 - 6W_0W_1W_2 + 2W_0W_1W_3)W_{n+3} + (W_3^3 + W_0^2W_1 - 2W_0W_2^2 - W_1W_2^2 + W_1^2W_3 - 4W_2W_3^2 + 4W_2^2W_3 - 2W_0W_1W_3 + 2W_0W_2W_3)W_{n+2} + (W_2^3 - W_0W_1^2 + W_0^2W_2 + W_0W_3^2 + 2W_1^2W_3 - 2W_0W_2W_3 - 2W_1W_2W_3)W_{n+1} + (W_1^3 + 2W_2^3 + W_0^2W_3 + W_1W_3^2 - W_2^2W_3 - 2W_0W_1W_2 - 2W_1W_2W_3)W_n$

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times P_{n+5} + b \times P_{n+4} + c \times P_{n+3} + d \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a\times P_5+b\times P_4+c\times P_3+d\times P_2 \\ W_1 &= a\times P_6+b\times P_5+c\times P_4+d\times P_3 \\ W_2 &= a\times P_7+b\times P_6+c\times P_5+d\times P_4 \\ W_3 &= a\times P_8+b\times P_7+c\times P_6+d\times P_5 \end{aligned}$$

we find that $a=4W_2-W_0-2W_3, b=4W_0-W_1-8W_2+4W_3, c=4W_1-4W_0-W_2, d=4W_2-4W_1-W_3$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 17. *The following equalities are true:*

(a) $22W_n=-(3W_0+2W_1+16W_2-10W_3)C_{n+5}-(4W_0-W_1-30W_2+16W_3)C_{n+4}+(16W_0-4W_1+W_2-2W_3)C_{n+3}+(2W_0+16W_1-4W_2-3W_3)C_{n+2}.$

(b) $22W_n=-(10W_0+3W_1+2W_2-4W_3)C_{n+4}+(16W_0-4W_1+W_2-2W_3)C_{n+3}+(2W_0+16W_1-4W_2-3W_3)C_{n+2}+(3W_0+2W_1+16W_2-10W_3)C_{n+1}.$

(c) $22W_n=-(4W_0+10W_1+3W_2-6W_3)C_{n+3}+(2W_0+16W_1-4W_2-3W_3)C_{n+2}+(3W_0+2W_1+16W_2-10W_3)C_{n+1}+(10W_0+3W_1+2W_2-4W_3)C_n.$

(d) $22W_n=-(6W_0+4W_1+10W_2-9W_3)C_{n+2}+(3W_0+2W_1+16W_2-10W_3)C_{n+1}+(10W_0+3W_1+2W_2-4W_3)C_n+(4W_0+10W_1+3W_2-6W_3)C_{n-1}.$

(e) $22W_n=-(9W_0+6W_1+4W_2-8W_3)C_{n+1}+(10W_0+3W_1+2W_2-4W_3)C_n+(4W_0+10W_1+3W_2-6W_3)C_{n-1}+(6W_0+4W_1+10W_2-9W_3)C_{n-2}.$

(f) $(W_0+W_1+W_2-W_3)(W_0^3+W_1^3+W_2^3-W_3^3+W_0W_1^2-W_0^2W_1+3W_0W_2^2-W_0^2W_2-W_0W_3^2+W_1W_2^2-W_0^2W_3-W_1^2W_2-W_1W_3^2-3W_1^2W_3+5W_2W_3^2-7W_2^2W_3-4W_0W_1W_2+4W_0W_1W_3+4W_1W_2W_3)C_n=-2(W_0^3+2W_1^3+4W_2^3-2W_3^3-2W_0^2W_1+5W_0W_2^2+2W_1W_2^2-W_1^2W_2-2W_1^2W_3+9W_2W_3^2-12W_2^2W_3-6W_0W_1W_2+6W_0W_1W_3-4W_0W_2W_3)W_{n+3}-2(-2W_2^3+3W_3^3+2W_0W_1^2+3W_0^2W_1-6W_0W_2^2-2W_0^2W_2-2W_0W_3^2-3W_1W_2^2-W_1^2W_3-12W_2W_3^2+12W_2^2W_3-6W_0W_1W_3+10W_0W_2W_3+4W_1W_2W_3)W_{n+2}-2(-2W_1^3-W_2^3-3W_0W_1^2+3W_0^2W_2+3W_0W_3^2-2W_0^2W_3-2W_1W_3^2+6W_1^2W_3+2W_2^2W_3+4W_0W_1W_2-6W_0W_2W_3-2W_1W_2W_3)W_{n+1}+2(2W_0^3+W_1^3+2W_2^3+2W_0W_2^2-3W_0^2W_3-2W_1^2W_2-3W_1W_3^2+2W_2W_3^2-5W_2^2W_3-6W_0W_1W_2+4W_0W_1W_3+6W_1W_2W_3)W_n$

Now, we give a few basic relations between $\{P_n\}$ and $\{C_n\}$.

Lemma 18. *The following equalities are true:*

$$
\begin{aligned}
22P_n &= 6C_{n+5}-3C_{n+4}-10C_{n+3}-4C_{n+2},\\
22P_n &= 9C_{n+4}-10C_{n+3}-4C_{n+2}-6C_{n+1},\\
22P_n &= 8C_{n+3}-4C_{n+2}-6C_{n+1}-9C_n,\\
22P_n &= 12C_{n+2}-6C_{n+1}-9C_n-8C_{n-1},\\
22P_n &= 18C_{n+1}-9C_n-8C_{n-1}-12C_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
C_n &= -4P_{n+5}+14P_{n+4}-12P_{n+3},\\
C_n &= 6P_{n+4}-12P_{n+3}+4P_{n+1},\\
C_n &= 4P_{n+1}-6P_n.
\end{aligned}
$$

5 Relations Between Special Numbers

In this section, we present identities on Pierre, Pierre-Lucas numbers and Tribonacci, Tribonacci-Lucas numbers. We know that

$$
\begin{aligned}
2P_n &= T_{n+2}+T_n-1,\\
C_n &= -T_{n+2}+4T_{n+1}-T_n+1,\\
C_n &= K_n+1,\\
22P_n &= 4K_{n+2}+2K_{n+1}-K_n-11.
\end{aligned}
$$

Note also that from Lemma 16 and Lemma 17, we have the formulas of W_n as

$$
\begin{aligned}
W_n &= (2W_1-W_2)P_{n+3}+(4W_2-4W_1-W_3)P_{n+2}+(W_0-4W_2+2W_3)P_{n+1}+(W_1-2W_0)P_n,\\
22W_n &= -(4W_0+10W_1+3W_2-6W_3)C_{n+3}+(2W_0+16W_1-4W_2-3W_3)C_{n+2}\\
&\quad +(3W_0+2W_1+16W_2-10W_3)C_{n+1}+(10W_0+3W_1+2W_2-4W_3)C_n.
\end{aligned}
$$

Using the above identities, we obtain the relation of generalized Pierre numbers and Tribonacci, Tribonacci-Lucas numbers in the following forms:

Lemma 19. *For all integers n, we have the following identities:*

(a) $2W_n=(3W_2-W_1-W_0-W_3)T_{n+2}+2(W_0-2W_2+W_3)T_{n+1}+(3W_1-W_0-3W_2+W_3)T_n-W_3+W_2+W_1+W_0.$

(b) $22W_n=(6W_1-2W_0-7W_2+3W_3)K_{n+2}-(W_0+8W_1-13W_2+4W_3)K_{n+1}+(6W_0-7W_1-W_2+2W_3)K_n+11(W_0+W_1+W_2-W_3).$

6 On the Recurrence Properties of Generalized Pierre Sequence

Taking $r = 2, s = 0, t = 0, u = -1$ in Theorem 5, we obtain the following Proposition.

Proposition 20. *For $n \in \mathbb{Z}$, generalized Pierre numbers (the case $r = 2, s = 0, t = 0, u = -1$) have the following identity:*

$$W_{-n} = \frac{1}{6}(-6W_{3n} + 6C_nW_{2n} - 3C_n^2W_n + 3C_{2n}W_n + W_0C_n^3 + 2W_0C_{3n} - 3W_0C_nC_{2n}).$$

From the above Proposition 20 (or by taking $G_n = P_n$ and $H_n = C_n$ in (1.10) and (1.11) respectively), we have the following corollary which gives the connection between the special cases of generalized Pierre sequence at the positive index and the negative index: for Pierre and Pierre-Lucas and Pierre numbers: take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4$ and take $W_n = C_n$ with $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8,$ respectively. Note that in this case $H_n = C_n$.

Corollary 21. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Pierre sequence:*

$$P_{-n} = \frac{1}{2}(-2P_{3n} + 2C_nP_{2n} - C_n^2P_n + C_{2n}P_n).$$

(b) *Pierre-Lucas sequence:*

$$C_{-n} = \frac{1}{6}\left(C_n^3 + 2C_{3n} - 3C_{2n}C_n\right).$$

We can also present the formulas of P_{-n} and C_{-n} in the following forms.

Corollary 22. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) $P_{-n} = -2(2P_{n+1} - 3P_n)^2P_n + (4P_{n+1} - 9P_n)P_{2n} + 2P_nP_{2n+1} - P_{3n}$.

(b) $P_{-n} = \frac{1}{2}(T_n^2 + T_{n-2}^2 + T_nT_{n+2} - 4T_nT_{n+1} + T_nT_{n-2} - 4T_{n-1}T_{n-2} + T_{2n} + T_{2n-4} - 1)$.

(c) $C_{-n} = \frac{1}{2}(K_n^2 - K_{2n} + 2)$.

Proof.

(a) By using the identity $C_n = 4P_{n+1} - 6P_n$ and Corollary 21, (or by using Corollary 6 (a)) we get

$$P_{-n} = -2(2P_{n+1} - 3P_n)^2P_n + (4P_{n+1} - 9P_n)P_{2n} + 2P_nP_{2n+1} - P_{3n}.$$

(b) Since $P_n = \frac{1}{2}(T_{n+2} + T_n - 1)$ and $T_{-n} = T_n^2 + T_{2n} + T_{n+2}T_n - 4T_{n+1}T_n$ (see, for example Soykan [24]), we get

$$P_{-n} = \frac{1}{2}(T_n^2 + T_{n-2}^2 + T_nT_{n+2} - 4T_nT_{n+1} + T_nT_{n-2} - 4T_{n-1}T_{n-2} + T_{2n} + T_{2n-4} - 1).$$

(c) Since $C_n = K_n + 1$ and $K_{-n} = \frac{1}{2}(K_n^2 - K_{2n})$ (see, for example Soykan [24]), we obtain

$$C_{-n} = \frac{1}{2}(K_n^2 - K_{2n} + 2). \square$$

7 Sum Formulas

The following Corollary gives sum formulas of Tribonacci and Tribonacci-Lucas numbers.

Corollary 23. *For $n \geq 0$, Tribonacci and Tribonacci-Lucas numbers have the following properties:*

(a)

- **(i)** $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} - T_{n+1} - 1)$.
- **(ii)** $\sum_{k=0}^{n} T_{2k} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1)$.
- **(iii)** $\sum_{k=0}^{n} T_{2k+1} = \frac{1}{2}(T_{2n+2} + T_{2n+1})$.

(b)

- **(i)** $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+3} - K_{n+1})$.
- **(ii)** $\sum_{k=0}^{n} K_{2k} = \frac{1}{2}(K_{2n+1} + K_{2n} + 2)$.
- **(iii)** $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} - 2)$.

Proof. It is given in Soykan [17, Corollary 2.3 and Corollary 2.4]. $\square$

The following Corollary presents sum formulas of Pierre and Pierre-Lucas numbers.

Corollary 24. *For $n \geq 0$, Pierre and Pierre-Lucas numbers have the following properties:*

(a)

- **(i)** $\sum_{k=0}^{n} P_k = \frac{1}{2}(2T_{n+2} + T_{n+1} + T_n - n - 3)$.
- **(ii)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{2}(T_{2n+2} + T_{2n+1} + T_{2n} - n - 2)$.
- **(iii)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{2}(2T_{2n+2} + 2T_{2n+1} + T_{2n} - n - 2)$.

(b)

- **(i)** $\sum_{k=0}^{n} C_k = \frac{1}{2}(K_{n+2} + K_n + 2(n+1))$.
- **(ii)** $\sum_{k=0}^{n} C_{2k} = \frac{1}{2}(K_{2n+1} + K_{2n} + 2(n+2))$.
- **(iii)** $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} + 2n)$.

Proof. The proof follows from Corollary 23 and the identities

$$\begin{aligned} P_n &= \frac{1}{2}(T_{n+2} + T_n - 1), \\ C_n &= K_n + 1. \ \square \end{aligned}$$

8 Matrices and Identities Related With Generalized Pierre Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & -W_{n-2} & -W_{n-1} & -W_n \\ W_n & -W_{n-3} & -W_{n-2} & -W_{n-1} \\ W_{n-1} & -W_{n-4} & -W_{n-3} & -W_{n-2} \\ W_{n-2} & -W_{n-5} & -W_{n-4} & -W_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 25. *For all integers $m, n,$ we have*

(a) $B_n = A^n$.

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Proof. Take $r = 2, s = 0, t = 0, u = -1$ **in Theorem 7.** □

Corollary 26. *For all integers n, we have the following formulas for the Pierre and Pierre-Lucas numbers.*

(a) *Pierre Numbers.*

$$A^n = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}.$$

(b) *Pierre-Lucas Numbers.*

$$A^n = \frac{1}{22}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= 8C_{n+4} - 4C_{n+3} - 6C_{n+2} - 9C_{n+1} \\ a_{21} &= 8C_{n+3} - 4C_{n+2} - 6C_{n+1} - 9C_{n} \\ a_{31} &= 8C_{n+2} - 4C_{n+1} - 6C_{n} - 9C_{n-1} \\ a_{41} &= 8C_{n+1} - 4C_{n} - 6C_{n-1} - 9C_{n-2} \end{aligned}$$

$$\begin{aligned} a_{12} &= -(8C_{n+1} - 4C_{n} - 6C_{n-1} - 9C_{n-2}) \\ a_{22} &= -(8C_{n} - 4C_{n-1} - 6C_{n-2} - 9C_{n-3}) \\ a_{32} &= -(8C_{n-1} - 4C_{n-2} - 6C_{n-3} - 9C_{n-4}) \\ a_{42} &= -(8C_{n-2} - 4C_{n-3} - 6C_{n-4} - 9C_{n-5}) \end{aligned}$$

$$\begin{aligned} a_{13} &= -(8C_{n+2} - 4C_{n+1} - 6C_{n} - 9C_{n-1}) \\ a_{23} &= -(8C_{n+1} - 4C_{n} - 6C_{n-1} - 9C_{n-2}) \\ a_{33} &= -(8C_{n} - 4C_{n-1} - 6C_{n-2} - 9C_{n-3}) \\ a_{43} &= -(8C_{n-1} - 4C_{n-2} - 6C_{n-3} - 9C_{n-4}) \end{aligned}$$

$$\begin{aligned} a_{14} &= -(8C_{n+3} - 4C_{n+2} - 6C_{n+1} - 9C_{n}) \\ a_{24} &= -(8C_{n+2} - 4C_{n+1} - 6C_{n} - 9C_{n-1}) \\ a_{34} &= -(8C_{n+1} - 4C_{n} - 6C_{n-1} - 9C_{n-2}) \\ a_{44} &= -(8C_{n} - 4C_{n-1} - 6C_{n-2} - 9C_{n-3}) \end{aligned}$$

Proof.

(a) It is given in Theorem 25 (a).

(b) Note that, from Lemma 18, we have

$$22P_n = 8C_{n+3} - 4C_{n+2} - 6C_{n+1} - 9C_n.$$

Using the last equation and (a), we get the required result. □

Using the above last Corollary and the identities

$$\begin{aligned} 2P_n &= T_{n+2} + T_n - 1, \\ 22P_n &= 4K_{n+2} + 2K_{n+1} - K_n - 11, \end{aligned}$$

we obtain the following formulas for Tribonacci and Tribonacci-Lucas numbers.

Corollary 27. *For all integers n, we have the following formulas for Tribonacci and Tribonacci-Lucas numbers.*

(a) *Tribonacci Numbers.*

$$A^n = \frac{1}{2}\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

where

$$\begin{aligned} b_{11} &= T_{n+2} + 2T_{n+1} + T_n - 1 \\ b_{21} &= T_{n+2} + T_n - 1 \\ b_{31} &= T_{n+2} - T_n - 1 \\ b_{41} &= -T_{n+2} + 2T_{n+1} + T_n - 1 \end{aligned}$$

$$\begin{aligned} b_{12} &= T_{n+2} - 2T_{n+1} - T_n + 1 \\ b_{22} &= -T_{n+2} + 2T_{n+1} - T_n + 1 \\ b_{32} &= -T_{n+2} + 3T_n + 1 \\ b_{42} &= 3T_{n+2} - 4T_{n+1} - 3T_n + 1 \end{aligned}$$

$$\begin{aligned} b_{13} &= -T_{n+2} + T_n + 1 \\ b_{23} &= T_{n+2} - 2T_{n+1} - T_n + 1 \\ b_{33} &= -T_{n+2} + 2T_{n+1} - T_n + 1 \\ b_{43} &= -T_{n+2} + 3T_n + 1 \end{aligned}$$

$$\begin{aligned} b_{14} &= -T_{n+2} - T_n + 1 \\ b_{24} &= -T_{n+2} + T_n + 1 \\ b_{34} &= T_{n+2} - 2T_{n+1} - T_n + 1 \\ b_{44} &= -T_{n+2} + 2T_{n+1} - T_n + 1 \end{aligned}$$

(b) *Tribonacci-Lucas Numbers.*

$$A^n = \frac{1}{22}\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

where

$$\begin{aligned} c_{11} &= 6K_{n+2} + 3K_{n+1} + 4K_n - 11 \\ c_{21} &= 4K_{n+2} + 2K_{n+1} - K_n - 11 \\ c_{31} &= -K_{n+2} + 5K_{n+1} + 3K_n - 11 \\ c_{41} &= 3K_{n+2} - 4K_{n+1} + 2K_n - 11 \end{aligned}$$

$$c_{12} = -3K_{n+2} + 4K_{n+1} - 2K_n + 11$$
$$c_{22} = -2K_{n+2} - K_{n+1} + 6K_n + 11$$
$$c_{32} = 6K_{n+2} - 8K_{n+1} - 7K_n + 11$$
$$c_{42} = -7K_{n+2} + 13K_{n+1} - K_n + 11$$

$$c_{13} = K_{n+2} - 5K_{n+1} - 3K_n + 11$$
$$c_{23} = -3K_{n+2} + 4K_{n+1} - 2K_n + 11$$
$$c_{33} = -2K_{n+2} - K_{n+1} + 6K_n + 11$$
$$c_{43} = 6K_{n+2} - 8K_{n+1} - 7K_n + 11$$

$$c_{14} = -4K_{n+2} - 2K_{n+1} + K_n + 11$$
$$c_{24} = K_{n+2} - 5K_{n+1} - 3K_n + 11$$
$$c_{34} = -3K_{n+2} + 4K_{n+1} - 2K_n + 11$$
$$c_{44} = -2K_{n+2} - K_{n+1} + 6K_n + 11$$

Next, we present an identity for W_{n+m}.

Theorem 28. *For all integers* m, n, *we have*

$$W_{n+m} = W_n P_{m+1} - W_{n-1} P_{m-2} - W_{n-2} P_{m-1} - W_{n-3} P_m. \tag{8.1}$$

Proof. Take $r = 2, s = 0, t = 0, u = -1$ **in Theorem 8.** □

As particular cases of the above theorem, we give identities for P_{n+m} **and** C_{n+m}**.**

Corollary 29. *For all integers* m, n, *we have*

$$\begin{aligned} P_{n+m} &= P_n P_{m+1} - P_{n-1} P_{m-2} - P_{n-2} P_{m-1} - P_{n-3} P_m, \\ C_{n+m} &= C_n P_{m+1} - C_{n-1} P_{m-2} - C_{n-2} P_{m-1} - C_{n-3} P_m. \end{aligned}$$

Taking $m = n$ **in the last corollary, we obtain the following identities:**

$$\begin{aligned} P_{2n} &= P_n P_{n+1} - 2P_{n-1} P_{n-2} - P_{n-3} P_n, \\ C_{2n} &= C_n P_{n+1} - C_{n-1} P_{n-2} - C_{n-2} P_{n-1} - C_{n-3} P_n. \end{aligned}$$

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