Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 6, Issue 2
Published online: January 2, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

The Signed Domination Number of Cartesian Products of Directed Cycles

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Abstract: Let D be a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$. A function $f:V(D)\rightarrow\{-1,1\}$ is called a signed dominating function (SDF) if $f(N_D^-[v])\geq 1$ for each vertex $v\in V$. The weight $w(f)$ of f is defined by $\sum_{v\in V} f(v)$. The signed domination number of a digraph D is $\gamma_s(D)$ = $min\{w(f) : f \text{ is an SDF of } D\}$. Let $C_m\times C_n$ denotes the Cartesian product of directed cycles of length m and n. In this paper, we determine the exact value of signed domination number of some classes of Cartesian product of directed cycles. In particular, we prove that: (a) $\gamma_s(C_3\times C_n) = n$ if $n \equiv 0 \pmod 3$, otherwise $\gamma_s(C_3\times C_n) = n + 2$. (b) $\gamma_s(C_4\times C_n) = 2n$. (c) $\gamma_s(C_5\times C_n) = 2n$ if $n \equiv 0 \pmod{10}$, $\gamma_s(C_5\times C_n) = 2n + 1$ if $n \equiv 3, 5, 7 \pmod{10}$, $\gamma_s(C_5\times C_n) = 2n + 2$ if $n \equiv 2, 4, 6, 8 \pmod{10}$, $\gamma_s(C_5\times C_n) = 2n + 3$ if $n \equiv 1,9 \pmod{10}$. (d) $\gamma_s(C_6\times C_n) = 2n$ if $n \equiv 0 \pmod 3$, otherwise $\gamma_s(C_6\times C_n) = 2n + 4$. (e) $\gamma_s(C_7\times C_n) = 3n$.

Keywords: *Directed graph, Directed cycle, Cartesian product, Signed dominating function, Signed domination number.*

1. Introduction

Throughout this paper, let D be a finite simple directed graph with the vertex set $V(D)$ and the arc set $A(D)$ (briefly V and A). If uv is an arc of D, then we also write $u\rightarrow v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex $v\in V$ let $N_D^+(v)$ and $N_D^-(v)$ denote the set of out-neighbors and in-neighbors of v, respectively. We write $d_D^+(v)=|N_D^+(v)|$ and $d_D^-(v)=|N_D^-(v)|$ for the outdegree and indegree of v in D, respectively (shortly $d^+(v)$, $d^-(v)$). A digraph D is r-regular if $d_D^+(v)=d_D^-(v)=r$ for any vertex $v\in D$. Let $N_D^+[v]=N_D^+(v)\cup\{v\}$ and $N_D^-[v]=N_D^-(v)\cup\{v\}$. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. For a real-valued function $f:V(D)\rightarrow R$ the weight of f is $w(f)=\sum_{v\in V} f(v)$, and for $S\subseteq V$, we define $f(S)=\sum_{v\in S} f(v)$, so $w(f)=f(V)$. Let $k\geq 1$ be an integer and let D be a digraph such that $\delta^-(D) \geq k - 1$. A *signed k-dominating function* ($SkDF$) of D is a function $f:V\rightarrow\{-1,1\}$ such that $f(N_D^-[v])\geq k$ for every vertex $v\in V$ (briefly $f[v] \geq k$). The *signed k-domination number* of a digraph D is $\gamma_{ks}(D) = \min\{w(f) : f \text{ is } SkDF \text{ of } D\}$. In particular, when k = 1, we get a definition of the *signed dominating function* and the *signed domination number*, i.e., $\gamma_s(D) = \gamma_{1s}(D)$. A signed dominating function of weight $\gamma_s(D)$ is defined a $\gamma_s(D)$-*function*. Consult [7] for the notation and terminology which are not defined here.

The Cartesian product $D_1 \times D_2$ of two digraphs D_1 and D_2 is the digraph with vertex set $V(D_1 \times D_2) = V(D_1) \times V(D_2)$ and $((u_1,u_2),(v_1,v_2)) \in A(D_1 \times D_2)$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in A(D_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in A(D_1)$.

The vertices of a directed cycle C_n are always denoted by the integers $\{1, 2, \ldots, n\}$, considered modulo n. The *i*th row of $V(C_m \times C_n)$ is $R_i = \{(i, j) : j = 1, 2, \ldots, n\}$ and the *j*th column $K_j = \{(i, j) : i = 1, 2, \ldots, m\}$. For any vertex $(i, j) \in V(C_m \times C_n)$, always we have the indices i and j are reduced modulo m and n, respectively. If f is a signed dominating function for $C_m \times C_n$, then we denote $f(K_j) = \sum_{i=1}^{m} f(i, j)$ of the weight of a column K_j and put $s_j = f(K_j)$. The sequence $(s_1, s_2, \ldots, s_n)$ is called a signed dominating function sequence corresponding to *f*.

In the past few years, several types of domination problems in graphs have been studied [2-4, 6, 10], most of those belonging to the vertex domination. In 1995, Dunbar et al. [4], have introduced the concept of signed domination number of an undirected graph. Haas and Wexler in [5], established a sharp lower bound on the signed domination number of a general graph with a given minimum and maximum degree and also of some simple grid graph. Zelinka [11] initiated the study of the signed domination numbers of digraphs. He studied the signed domination number of digraphs for which the in-degrees does not exceed 1, as well as for acyclic tournaments and the circulant tournaments. Karami et al. [8] were established lower and upper bounds of the signed domination number of digraphs. Atapour et al. [1], presented some sharp lower bounds on the signed k-domination number of digraphs. Shaheen [9] calculated the signed domination numbers of Cartesian product of two paths $P_m \times P_n$ for m = 2, 3, 4, 5, 6, 7 and arbitrary n. In this paper, we study the Cartesian product $C_m \times C_n$ of C_m and C_n for m, n $\geq$ 3. We mainly determine the exact values of $\gamma_s(C_3 \times C_n)$, $\gamma_s(C_4 \times C_n)$, $\gamma_s(C_5 \times C_n)$, $\gamma_s(C_6 \times C_n)$ and $\gamma_s(C_7 \times C_n)$.

Let us introduce a definition. Suppose that *f* is a signed dominating function for $C_m \times C_n$, and assume that $1 \leq j, h \leq n$. We say that the *h*th column in $C_m \times C_n$ is an *t-shift* of the *j*th column if $f(i, j) = f(i + t, h)$ for each vertex $(i, j) \in K_j$, where the indices i, t, i + t are taken modulo m and j, h are taken modulo n.

Theorem 1.1(Zelinka [11]). Let D be a directed cycle or path with n vertices. Then $\gamma_s(D) = n$.

Lemma 1.2 (Zelinka [11]). Let D be a digraph with n vertices. Then $\gamma_s(D) \equiv n \pmod 2$.

Theorem 1.3 (Karami et al. [8]). Let D be a digraph of order n and let k be a nonnegative integer such that $d^-(v) \geq k$ for each $v \in V(D)$. Then

$$\gamma_S(D) \geq n \frac{1 + k + 2\left\lceil \frac{k}{2} \right\rceil - \Delta}{1 - k + \Delta}$$

Corollary 1.4 (Karami et al. [8]). Let D be a digraph of order n in which $d^+(v) = d^-(v) = k$ for each $v \in V$, where k is a nonnegative integer. Then $\gamma_S(D) \geq \frac{n}{1+k}$.

Theorem 1.5 (Atapour et al. [1]). Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^- \geq k - 1$. Then

$$\gamma_{kS}(D) \geq n \frac{2\left\lceil \frac{\delta^- + k + 1}{2} \right\rceil - 1 - \Delta^+}{\Delta^+ + 1}.$$

2. Main results

In this section we calculate the signed domination number of the Cartesian product of two directed cycles C_m and C_n for m = 3, 4, 5, 6, 7 and arbitrary n. We should note that, for simplicity of drawing the

Cartesian products of two directed cycles $C_m \times C_n$, we do not draw the arcs from vertices in last column to vertices in first column and the arcs from vertices in last row to vertices in first row.

Remark 2.1: Let f be a $\gamma_s(C_m \times C_n)$-function. Then $f[(r, s)] \geq 1$ for each $1 \leq r \leq m$ and each $1 \leq s \leq n$. Since $C_m \times C_n$ is 2-regular, it follows from $f((i, j)) = -1$ that $f((i\pm1, j)) = f((i, j\pm1)) = 1$ because $f[(i, j)] \geq 1$, $f((i+1, j-1)) = 1$ because $f[(i+1, j)] \geq 1$ and $f((i-1, j+1)) = 1$ because $f[(i, j+1)] \geq 1$. On the other hand, if $f((i\pm1, j)) = f((i, j\pm1)) = 1$, $f((i+1, j-1)) = 1$ and $f((i-1, j+1)) = 1$, then we must have $f((i, j)) = -1$ since f is a minimum signed dominating function.

Remark 2.2. Since the case $f((i, j)) = f((i+1, j)) = -1$ is not possible, we get $s_j \geq 0$. Furthermore, s_j is odd if m is odd and even when m is even.

Theorem 2.1. $\gamma_s(C_3 \times C_n) = \begin{cases} n & \text{if } n \equiv 0(\text{mod}(3)), \\ n+2 & \text{otherwise.} \end{cases}$

Proof. Corollary 1.3, implies that $\gamma_s(C_3 \times C_n) \geq n$. (1)

In any case we cannot put more than -1 in each column. We distinguish two cases:

Case 1. $n \equiv 0(\text{mod } 3)$: We define a function $f((i, j)) = -1$ where $i \equiv j(\text{mod } 3)$ for $j = 1, \ldots, n$ and $f((i, j)) = 1$ otherwise. This is a signed dominating function SDF for $C_3 \times C_n$. Furthermore, $s_j = \sum_{i=1}^{3} f((i, j)) = 1$ which means that $\gamma_s(C_3 \times C_n) \leq n$. This together with (1) imply $\gamma_s(C_3 \times C_n) = n$.

Case 2. $n \equiv 1, 2(\text{mod } 3)$: The same function defined in the previous case with $j < n$, then $s_j = 1$ for $j = 1, 2, \ldots, n-1$ and let $f((i, n)) = 1$ for $i = 1, 2, 3$. Then f is SDF of $C_3 \times C_n$ with $w(f) = n+2$. Without loss of generality, we can assume $f((1,1)) = -1$. By Remark 2.1, we have $f((2, 1)) = f((3,1)) = f((1,2)) = f((3,2)) = 1$ and we can only put $f((2, 2)) = -1$. By similar arguments $f((1,3)) = f((3,2)) = 1$ and $f((3,3)) = -1$. We deduce that $f((1, 1)) = f((2,2)) = f((3,3)) = f((1,4)) = f((2,5)) = f((3,6)) = \ldots = f((1,3k+1)) = f((2, 3k+2)) = f((3, 3k+3)) = \ldots = -1$.

If $n \equiv 1(\text{mod } 3)$, then K_n is 0-shift of K_1 and this implies that $f((1, n)) = -1$ and $f[(1,1)] = -1$, this is a contradiction. So, we have $f((1, n)) = 1$. In the same time $f((3,n-1)) = -1$, then $f((3,n)) = 1$ and $f((2,n)) = 1$ (otherwise $f[(2, 1)] = -1$), which implies that $s_n = 3$. Hence,

$w(f) \geq \sum_{j=1}^{n-1} s_j + s_n = n-1+3 = n+2$. We conclude that $\gamma_s(C_3 \times C_n) = n+2$.

If $n \equiv 2$ (mod 3), by similar arguments to the case $n \equiv 1(\text{mod } 3)$, is the required (with notice that K_n is 1-shift of K_1). □

Theorem 2.2. $\gamma_s(C_4 \times C_n) = 2n$.

Proof. We define a signed dominating function f as follows:

$f((i, j)) = -1$ where $i \equiv j(\text{mod } 4)$ for $j = 1, \ldots, n$, and $f((i, j)) = 1$ otherwise.

$f_{n-3}((3, n-3)) = f_{n-2}((4, n-2)) = f_{n-1}((1, n-1)) = f_n((3, n)) = -1$, and $f((i, j)) = 1$ otherwise for $j = n-3, n-2, n-1, n$. Obviously,

f is a SDF of $C_4 \times C_n$ for $n \equiv 0, 3(\text{mod } 4)$. $\{f \setminus \{f(K_n)\} \cup \{f_n\}$ is a SDF for $C_4 \times C_n$ when $n \equiv 2(\text{mod } 4)$.

$\{f \setminus \{f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\} \cup \{f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$ is a SDF for $C_4 \times C_n$ when $n \equiv 1(\text{mod } 4)$.

We have $s_j = \sum_{i=1}^{4} f((i, j)) = 2$ for $j = 1, \ldots, n$, and $w(f) = 2n$. Therefore,

$$\gamma_s(C_4 \times C_n) \leq 2n. \quad (2)$$

Let f' is a SDF of $C_4 \times C_n$. By Remark 2.1, the case $f'((i, j)) = f'((i+1, j)) = -1$ is not exist. This implies that, for any column K_j there are two cases:

Case 1. In K_j we have $f'((i, j)) = f'((i+2, j)) = -1$, and $f'((i+1, j)) = f'((i+3, j)) = 1$. Then $f'((i, j \pm 1)) = 1$ for $i = 1,2,3,4$. Which leads, if $s'_j = 0$ then $s'_{j-1} = s'_{j+1} = 4$. So, $\gamma_s(C_4 \times C_n) \geq 2n$.

Case 2. In K_j we have $f'((i, j)) = -1$ and $f'((i+1, j)) = f'((i+2, j)) = f'((i+3, j)) = 1$. Then $f'((i, j+1)) = f'((i-1, j+1)) = 1$. By Remark 2.1, only one of $f'((i+1, j+1))$ or $f'((i+2, j+1))$ is equals -1. We conclude that each column can not including more than one vertex which gets -1 and $s'_j \geq 2$ for $j = 1, 2, \ldots, n$. Furthermore, $w(f') = \sum_{j=1}^{n} s'_j \geq 2n$. Applying (2), together with the Cases 1 and 2, we get $\gamma_s(C_4 \times C_n) = 2n$. □

Theorem 2.3.

$$\gamma_s(C_5 \times C_n) = \begin{cases} 2n & \text{if } n \equiv 0 \pmod{10}, \\ 2n+1 & \text{if } n \equiv 3,5,7 \pmod{10}, \\ 2n+2 & \text{if } n \equiv 2,4,6,8 \pmod{10}, \\ 2n+3 & \text{if } n \equiv 1,9 \pmod{10}. \end{cases}$$

Proof. We define a signed dominating function f as follows:

$f((4i-3, 2j-1)) = -1$ for $1 \leq j \leq \lceil n/2 \rceil$ and $i \equiv j \pmod 5$,

$f((4i-2, 2j)) = f(4i, 2j) = -1$ for $1 \leq j \leq \lfloor n/2 \rfloor$ and $i \equiv j \pmod 5$, and

$f((i, j)) = 1$ otherwise.

By define f, we have $s_j = 3$ for j is odd and $s_j = 1$ for j is even. Also, f is a SDF for $C_5 \times C_n$ when $n \equiv 0,3,5,7 \pmod{10}$. And f is a SDF of the vertices of $K_2, \ldots, K_n$, when $n \equiv 1,2,4,6,8,9 \pmod{10}$.

Now, let us a functions $f_1((4, n)) = -1$ and $f_1((i, n)) = 1$ for $i = 1, 2, 3, 5$. $f_2((3, n)) = -1$ and $f_2((i, n)) = 1$ for $i = 1, 2, 4, 5$. $f_3((5, n)) = -1$ and $f_3((i, n)) = 1$ for $i = 1, 2, 3, 4$. And $f_4((i, n)) = 1$ for $i = 1, 2, 3, 4, 5$. We note:

$\{f \setminus f(K_n)\} \cup f_1$ is a SDF of $C_5 \times C_n$ when $n \equiv 2, 8 \pmod{10}$.

$\{f \setminus f(K_n)\} \cup f_2$ is a SDF of $C_5 \times C_n$ when $n \equiv 4 \pmod{10}$.

$\{f \setminus f(K_n)\} \cup f_3$ is a SDF of $C_5 \times C_n$ when $n \equiv 6 \pmod{10}$.

$\{f \setminus f(K_n)\} \cup f_4$ is a SDF of $C_5 \times C_n$ when $n \equiv 1, 9 \pmod{10}$. For an illustration $\gamma_s(C_5 \times C_{11})$, see Figure 1. Also,

$$\begin{aligned} &\gamma_s(C_5 \times C_n) \leq 2n, \text{ if } n \equiv 0 \pmod{10}, \\ &\gamma_s(C_5 \times C_n) \leq 2n+1, \text{ if } n \equiv 3, 5, 7 \pmod{10}, \\ &\gamma_s(C_5 \times C_n) \leq 2n+2 \text{ for } n \equiv 2, 4, 6, 8 \pmod{10}, \\ &\gamma_s(C_5 \times C_n) \leq 2n+3 \text{ for } n \equiv 1, 9 \pmod{10}. \end{aligned} \tag{3}$$

By Remark 2.2, for any minimum signed dominating function f of $C_5 \times C_n$ with signed dominating function sequence $(s_1, \ldots, s_n)$, we have $s_j \geq 1$. Furthermore $s_j = 1, 3$ or 5 for $j = 1, \ldots, n$. Also, if $s_j = 1$ then $s_{j-1}, s_{j+1} \geq 3$. This implies that $w(f) = \sum_{j=1}^{n} s_j \geq 2n$ for n is even, and $w(f) = \sum_{j=1}^{n} s_j \geq 2n+1$ for n is odd. Thus with (3), gets

$\gamma_s(C_5 \times C_n) = 2n$ if $n \equiv 0 \pmod{10}$ and $\gamma_s(C_5 \times C_n) = 2n+1$ if $n \equiv 3, 5, 7 \pmod{10}$.

For n ≡ 1, 9(mod 10).

We will show $\gamma_s(C_5\times C_n) \geq 2$ n + 3 when n ≡ 1, 9(mod 10). We consider the case n ≡ 1(mod 10), and the case n ≡ 9(mod 10) is similar to it.

Let us $2n + 1 \leq \gamma_s(C_5\times C_n) \leq 2n + 3$. By Lemma 1.2, $\gamma_s(C_5\times C_n) \equiv 5n \pmod 2$, this implies that $\gamma_s(C_5\times C_n) = 2n + 1$ or $\gamma_s(C_5\times C_n) = 2n + 3$.

We know that s_j =1, 3 or 5 and $s_j = s_{j+1} = 1$ is not possible. If there is one column K_j with $s_j = 5$, then $w(f) = \sum_{j=1}^{n} s_j \geq 2n+3$. By using (3) the case is finished.

Assume that $s_j < 5$ for all j, then there are only two values of s_j its 1 and 3. Suppose that $\gamma_s(C_5\times C_n) = 2n + 1$. Then there are (n + 1)/2 terms of $s_j = 3$ and (n - 1)/2 terms of $s_j = 1$. Which implies that, there are $s_j = s_{j+1} = 3$. Without loss of generality, we can assume that $s_1 = s_n = 3$. Then we gets the form $s_j = 3$ where j is odd, and $s_j = 1$ where j is even. So, let us $s_1 = 3$ and $f((1, 1)) = -1$, then $s_2 = 1$ and $f((2, 2)) = f((4, 2)) = -1$. Also, $s_3 = 3$ and $f((5, 3)) = -1$, $s_4 = 1$ and $f((1, 4)) = f((3, 4)) = -1$. We deduce that each column K_j is 4-shift of K_{j-2}. Furthermore, K_n is 0-shift of K_1 {4(n -1)/2 = 2n -2 ≡ 0(mod 5)}, i.e. $f((1, n)) = -1$, and this is a contradiction. Therefore $\gamma_s(C_5\times C_n) > 2n + 1$ and $\gamma_s(C_5\times C_n) = 2n + 3$.

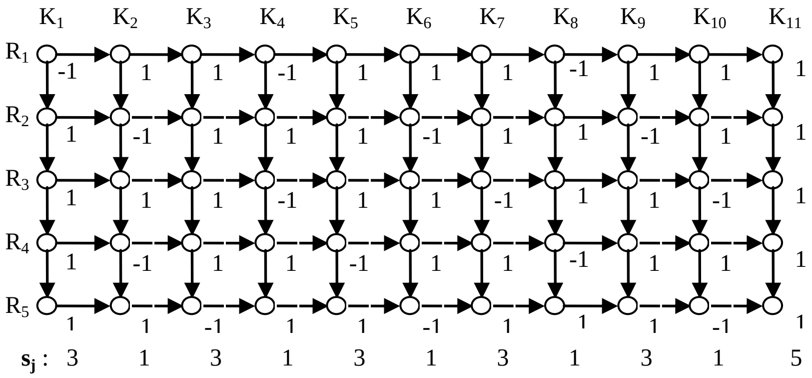

Figure 1. A signed dominating function of $C_5\times C_{11}$.

A corresponding matrix of a signed dominating function of $C_5\times C_{11}$

$$\begin{array}{c|ccccccccccc|}
 & K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 & K_{10} & K_{11} \\
R_1 & - & + & + & - & + & + & + & - & + & + & + \\
R_2 & + & - & + & + & + & - & + & + & - & + & + \\
R_3 & + & + & + & - & + & + & - & + & + & - & + \\
R_4 & + & - & + & + & - & + & + & - & + & + & + \\
R_5 & + & + & - & + & + & - & + & + & + & - & + \\
\mathbf{s_j}: & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 5
\end{array}$$

{Here, we must note that, for simplicity of drawing the Cartesian products of two directed cycles $C_m\times C_n$, we do not draw the arcs from vertices in last column to vertices in first column and the arcs from vertices in last row to vertices in first row. Also for each figure of $C_m\times C_n$, we replace it by a corresponding matrix by signs – and + which descriptions -1 and + 1 on figure of $f(C_m\times C_n)$, respectively}.

For n ≡ 2, 4, 6, 8(mod 10).

We will show $\gamma_s(C_5\times C_n) \geq 2n + 2$ when n ≡2, 4, 6, 8(mod 10). We study the case n ≡ 8(mod 10), the remained cases are similar to it.

Let n ≡ 8(mod 10). By Lemma 1.2, $\gamma_s(C_5\times C_n) \equiv 5n \pmod 2$, so $\gamma_s(C_5\times C_n) = 2n$ or $\gamma_s(C_5\times C_n) = 2n + 2$. Assume that $\gamma_s(C_5\times C_n) = 2n$. Then by the same argument similar to the case n ≡ 1(mod 10), we get $s_n = 1$. Furthermore, K_n is (4(n – 2)/2 = 4(10k + 8 – 2)/2 = 4(5k + 3) = 2-shift of K_2. This mining that $f((1, n)) = f((4, n)) = -1$, and this is a contradiction. Therefore $\gamma_s(C_5\times C_n) > 2n$, and by (3) is $\gamma_s(C_5\times C_n) = 2n + 2$. □

Theorem 2. 4.

$$\gamma_s(C_6 \times C_n) = \begin{cases} 2n & \text{if } n \equiv 0 \pmod 3, \\ 2n+4 & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proof. We define a signed dominating function f as follows:

f ((i, j)) = -1 and f ((i + 3, j)) = -1 for $1 \le j \le n$ and $i \equiv j \pmod 6$, and f ((i, j)) = 1 otherwise.

Also, f_1((i, n)) = 1 for i = 1, …, 6.

By define f, we have $s_j = 2$ for all $1 \le j \le n$. Notice that f is a SDF for $C_6 \times C_n$ where $n \equiv 0 \pmod 3$, and $\gamma_s(C_6 \times C_n) \le 2n$. Also, f is a SDF of the vertices of $K_2, \ldots, K_n$ when $n \equiv 1, 2 \pmod 3$. So, $\{f \setminus f(K_n)\} \cup f_1$ is a SDF for $C_6 \times C_n$ where $n \equiv 1, 2 \pmod 3$, and $\gamma_s(C_6 \times C_n) \le 2n + 4$. For an illustration $\gamma_s(C_6 \times C_8)$, see Figure 2.

By Remark 2.2, we have s_j = 0, 2, 4 or 6. If $s_j = 0$, then $s_{j-1} = s_{j+1} = 6$. Also, when $s_j = 2$ is $s_{j-1}, s_{j+1} \ge 2$. We deduce that $\gamma_s(C_6 \times C_n) = \sum_{j=1}^{n} s_j \ge 2n$. Hence, $\gamma_s(C_6 \times C_n) = 2n$ for $n \equiv 0 \pmod 3$.

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ \mathbf{s_j}: \end{array}
\begin{array}{c}
\begin{array}{cccccccc} K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 \end{array} \\
\begin{bmatrix}
- & + & + & - & + & + & - & + \\
+ & - & + & + & - & + & + & + \\
+ & + & - & + & + & - & + & + \\
- & + & + & - & + & + & - & + \\
+ & - & + & + & - & + & + & + \\
+ & + & - & + & + & - & + & +
\end{bmatrix} \\
\begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 \end{array}
\end{array}$$

Figure 2: A corresponding matrix of a signed dominating function of $C_6 \times C_8$.

For $n \equiv 1, 2 \pmod 3$. We will show that $\gamma_s(C_6 \times C_n) \ge 2n + 4$.

If $s_j = 0$ for some j, then $\sum_{j-1}^{j+1} s_j = 12$. Since $\sum_{j} s_j \ge 2j$ for $j \ge 2$, then $\gamma_s(C_6 \times C_n) \ge 2n + 4$. Assume that $s_j \ge 2$ for all j. If there is one $s_j = 6$ or two of s_j are equal 4, then gets the required. Now, assume that $s_j = 2$ for all j accept once which is equal 4, i.e. $\sum_{j=1}^{n} s_j = 2n + 2$. We prove the following claim:

Claim 2.1. If $s_j = \ldots = s_{j+k} = 2$ (for $k \ge 1$), then we have one possible of f is:

f ((i, j)) = f ((i + 3, j)) = -1 ⇔ f ((i +1, j +1)) = f ((i + 4, j +1)) = -1. Furthermore, each column K_j is 1-shift of K_{j-1}.

Proof of Claim 2.1. Since $s_j = \ldots = s_{j+k} = 2$ (for $k \ge 1$), we have each column include two vertices are assigned value -1. By Remark 2.2, we can assume that f((i, j)) = f((i + 2, j)) = -1, this implies that f((i -1, j +1)) = f((i, j +1)) = f((i +1, j +1)) = f((i +2, j +1)) = 1. Furthermore, at most one of the remaining vertices of K_{j+1} is assigned value -1. Which conclude that $s_{j+1} \ge 4$, and is a contradiction. The cases f((i, j)) = f((i +4, j)) = -1 and f((i, j)) = f((i +5, j)) = -1 are similar by symmetry to the cases f((i, j)) = f((i +2, j)) = -1 and f((i, j)) = f((i +1, j)) = -1, respectively. Thus, we left with one case which f ((i, j)) = f ((i +3, j)) = -1 ⇔ f ((i +1, j +1)) = f ((i +4, j +1)) = -1. Also, K_j is 1-shift of K_{j-1}. The proof of Claim 2.1 is complete.

By Claim 2.1, and without loss of generality, we can assume $s_1 = \ldots = s_{n-1} = 2$ and $s_n = 4$. Then K_{n-1} is (n -2)-shift of K_1. Let f((1, 1)) = f((4, 1)) = -1, we distinguish two cases:

If $n \equiv 1 \pmod 3$, then $f((3, n-1)) = f((6, n-1)) = -1$. This implies that $f((2, n)) = f((3, n)) = f((5, n)) = f((6, n)) = 1$. Since $s_n = 4$, we must have one of $f((1, n))$, $f((4, n))$ is equal -1. This is a contradiction, because $f((1, 1)) = f((4, 1)) = -1$.

If $n \equiv 2 \pmod 3$, then $f((1, n-1)) = f((4, n-1)) = -1$. By the same argument to above case, we get a contradiction, because $f((1, 1)) = f((4, 1)) = -1$.

From previous arguments, we conclude $\gamma_s(C_6 \times C_n) > 2n + 2$. By Lemma 1.2, $\gamma_s(C_6\times C_n) \equiv 6n \pmod 2$. So, $\gamma_s(C_6\times C_n) = 2n + 4$ when $n \equiv 1, 2 \pmod 3$. □

Theorem 2. 5. $\gamma_s(C_7\times C_n) = 3n$, where $n \geq 7$.

Proof. We define a signed dominating function f as follows:

$f((i, j)) = f((i+3, j)) = -1$ for $1 \leq j \leq n$ and $i \equiv j \pmod 7$, and $f((i, j)) = 1$ otherwise. Also, we define

$f_{n-4}((4, n-4)) = f_{n-4}((7, n-4)) = -1$, $f_{n-3}((2, n-3)) = f_{n-3}((5, n-3)) = -1$, $f_{n-2}((3, n-2)) = f_{n-2}((7, n-2)) = -1$, $f_{n-1}((1, n-1)) = f_{n-1}((5, n-1)) = -1$, $f_n((3, n)) = f_n((6, n)) = -1$ and $f_j((i, j)) = 1$ otherwise for $j = n-4, n-3, n-2, n-1, n$.

By define f and f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1} and f_n we have $s_j = 3$ for all $1 \leq j \leq n$. Notice that:

f is a SDF for $C_7\times C_n$ when $n \equiv 0, 3 \pmod 7$.

$\{f \backslash \{f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\} \cup \{f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$ is a SDF for $C_7\times C_n$ when $n \equiv 1 \pmod 7$. For an illustration $\gamma_s(C_7\times C_8)$, see Figure 3.

$\{f \backslash \{f(K_{n-1}) \cup f(K_n)\} \cup \{f_{n-1} \cup f_n\}$ is a SDF for $C_7\times C_n$ when $n \equiv 2 \pmod 7$.

$\{f \backslash \{f(K_{n-4}) \cup f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\} \cup \{f_{n-4} \cup f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$ is a SDF for $C_7\times C_n$ when $n \equiv 4 \pmod 7$.

$\{f \backslash \{f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\} \cup \{f_{n-2} \cup f_{n-1} \cup f_n\}$ is a SDF for $C_7\times C_n$ when $n \equiv 5 \pmod 7$.

$\{f \backslash \{f(K_n)\} \cup \{f_n\}$ is a SDF for $C_7\times C_n$ when $n \equiv 6 \pmod 7$.

In all the cases we have $\gamma_s(C_7\times C_n) \leq 3n$.

By Remark 2.2, we have $s_j = 1, 3, 5$ or 7. Also, if $s_j = 1$, then $s_{j-1}, s_{j+1} \geq 5$ and when $s_j = 3$, is $s_{j-1}, s_{j+1} \geq 3$. This implies that $\gamma_s(C_7 \times C_n) = \sum_{j=1}^{n} s_j \geq 3n$. So, we get $\gamma_s(C_7\times C_n) = 3n$. □

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 \end{array} \\
\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \end{array} &
\left[\begin{array}{cccccccc}
- & + & + & + & + & + & - & + \\
+ & - & + & + & - & + & + & + \\
+ & + & - & + & + & - & + & - \\
- & + & + & - & + & + & + & + \\
+ & - & + & + & - & + & - & + \\
+ & + & - & + & + & + & + & - \\
+ & + & + & - & + & - & + & +
\end{array}\right] \\
\mathbf{s_j}: & \begin{array}{cccccccc} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{array}
\end{array}
$$

Figure 3: A corresponding matrix of a signed dominating function of $C_7\times C_8$.

3. Conclusions

In this paper, we determined the exact value of the signed domination number of $C_m\times C_n$ for $m = 3, \ldots, 7$ and arbitrary n. By using same technique methods, our hope eventually lead to determination $\gamma_s(C_m\times C_n)$ for $m \geq 8$.

Based on the above (Remark 2.1 and Theorems 2.1, …, 2.5), also by the technique which used in this paper, we arrive to the following conjecture:

Conjecture 3. 1.

$$\gamma_s(C_m \times C_n) = \left\lceil \frac{m}{3} \right\rceil n \text{ when } m, n \equiv 0 \pmod 3), n \equiv 0 \pmod{2m} \text{ or } n \equiv 1 \pmod 3).$$

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