On The Diophantine Equation $\sum x_i^2 + a = y^2$ And $\sum x_i^3 + a = y^3$

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Abstract:

In this paper, the Diophantine equations $\sum x_i^2 + a = y^2$ and $\sum x_i^3 + a = y^3$ where $x_1 \neq x_2 \neq x_3 \neq \cdots$ and $a$ is a positive integer have been discussed for possible positive integral solutions.

Keywords: Diophantine equation and integral solution.

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1 Introduction:

Several researchers have discussed the non-linear Diophantine equations. Most famous non-linear Diophantine equations are Fermat’s Last Problem (1637) and Beal’s Conjecture (1993). Fermat’s Last Problem was solved by Andrew Wiles (1995). Beal’s Conjecture is still an open unsolved problem. Gandhi & Sarma (2013) attempted to disprove the Beal’s conjecture. Gregorio, L.T.D. (2013) presented a proof for the the Beal’s conjecture and a new proof for the Fermat’s last theorem.


In this paper, we have discussed the Diophantine equations $\sum x_i^2 + a = y^2$ and $\sum x_i^3 + a = y^3$ where $x_1 \neq x_2 \neq x_3 \neq \cdots$ and $a$ is a positive integer. The physical interpretation of first Diophantine equation is that some squares of different dimensions and a strip of dimension $a \times 1$ is equal to a big square while the physical interpretation of second Diophantine equation is that some cubes of different dimensions and solid strip of dimensions $a \times 1 \times 1$ is equal to a big cube.
2 Analysis: (A) Diophantine equation $\sum x_i^2 + a = y^2$: This Diophantine equation will be discussed for $a = 10, 20, 30, 40, 50$ and $60$.

(a) For $a = 10$ the given Diophantine equation becomes

$$\sum x_i^2 + 10 = y^2. \quad \ldots (1)$$

(i) If we take $x_1 = 3$, $x_2 = 9$ and $y = 10$ then equation (1) is satisfied. Thus $(x_1, x_2, y) = (3, 9, 10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 10 = y^2$.

(ii) If we take $x_1 = 1$, $x_2 = 2, x_3 = 3, x_4 = 5$ and $y = 7$ then equation (1) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (1, 2, 3, 5, 7)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 10 = y^2$.

(iii) If we take $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 5$ and $y = 8$ then equation (1) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2, 3, 4, 5, 8)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 10 = y^2$.

(b) For $a = 20$ the given Diophantine equation becomes

$$\sum x_i^2 + 20 = y^2. \quad \ldots (2)$$

(i) If we take $x_1 = 7, x_2 = 10$ and $y = 13$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (7, 10, 13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(ii) If we take $x_1 = 4, x_2 = 8$ and $y = 10$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (4, 8, 10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(iii) If we take $x_1 = 6, x_2 = 7, x_3 = 8$ and $y = 13$ then equation (2) is satisfied. Thus $(x_1, x_2, x_3, y) = (6, 7, 8, 13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(iv) If we take $x_1 = 6, x_2 = 13$ and $y = 15$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (6, 13, 15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(c) For $a = 30$ the given Diophantine equation becomes

$$\sum x_i^2 + 30 = y^2. \quad \ldots (3)$$

(i) If we take $x_1 = 3, x_2 = 5, x_3 = 6$ and $y = 10$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (3, 5, 6, 10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(ii) If we take $x_1 = 3, x_2 = 7, x_3 = 9$ and $y = 13$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (3, 7, 9, 13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(iii) If we take $x_1 = 6, x_2 = 7, x_3 = 9$ and $y = 14$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (6, 7, 9, 14)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$. 

(iv) If we take \(x_1 = 5\), \(x_2 = 7\), \(x_3 = 11\) and \(y = 15\) then equation (3) is satisfied. Thus \((x_1, x_2, x_3, y) = (5, 7, 11, 15)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + 30 = y^2\).

(d) For \(a = 40\) the given Diophantine equation becomes

\[
\sum x_i^2 + 40 = y^2. \quad \ldots (4)
\]

(i) If we take \(x_1 = 3\) and \(y = 7\) then equation (4) is satisfied. Thus \((x_1, y) = (3, 7)\) is the solution of the Diophantine equation \(x_1^2 + 40 = y^2\).

(ii) If we take \(x_1 = 9\) and \(y = 11\) then equation (8.4) is satisfied. Thus \((x_1, y) = (9, 11)\) is the solution of the Diophantine equation \(x_1^2 + 40 = y^2\).

(iii) If we take \(x_1 = 4\), \(x_2 = 5\) and \(y = 9\) then equation (4) is satisfied. Thus \((x_1, x_2, y) = (4, 5, 9)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + 40 = y^2\).

(iv) If we take \(x_1 = 2\), \(x_2 = 10\) and \(y = 12\) then equation (4) is satisfied. Thus \((x_1, x_2, y) = (2, 10, 12)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + 40 = y^2\).

(v) If we take \(x_1 = 2\), \(x_2 = 9\), \(x_3 = 10\) and \(y = 15\) then equation (4) is satisfied. Thus \((x_1, x_2, x_3, y) = (2, 9, 10, 15)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + 40 = y^2\).

(vi) If we take \(x_1 = 4\), \(x_2 = 5\), \(x_3 = 12\) and \(y = 15\) then equation (4) is satisfied. Thus \((x_1, x_2, x_3, y) = (4, 5, 12, 15)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + 40 = y^2\).

(vii) If we take \(x_1 = 6\), \(x_2 = 7\), \(x_3 = 10\) and \(y = 15\) then equation (4) is satisfied. Thus \((x_1, x_2, x_3, y) = (6, 7, 10, 15)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + 40 = y^2\).

(viii) If we take \(x_1 = 4\), \(x_2 = 13\) and \(y = 15\) then equation (4) is satisfied. Thus \((x_1, x_2, y) = (4, 13, 15)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + 40 = y^2\).

(ix) If we take \(x_1 = 2\), \(x_2 = 4\), \(x_3 = 14\) and \(y = 16\) then equation (4) is satisfied. Thus \((x_1, x_2, x_3, y) = (2, 4, 14, 16)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + 40 = y^2\).

(x) If we take \(x_1 = 4\), \(x_2 = 6\), \(x_3 = 8\), \(x_4 = 10\) and \(y = 16\) then equation (8.4) is satisfied. Thus \((x_1, x_2, x_3, x_4, y) = (4, 6, 8, 10, 16)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + x_3^2 + x_4^2 + 40 = y^2\).

(e) For \(a = 50\) the given Diophantine equation becomes

\[
\sum x_i^2 + 50 = y^2. \quad \ldots (5)
\]

(i) If we take \(x_1 = 5\), \(x_2 = 11\) and \(y = 14\) then equation (5) is satisfied. Thus \((x_1, x_2, y) = (5, 11, 14)\) is the solution of the Diophantine equation \(x_1^2 + x_2^2 + 50 = y^2\).

(f) For \(a = 60\) the given Diophantine equation becomes

\[
\sum x_i^2 + 60 = y^2. \quad \ldots (6)
\]
(i) If we take \( x_1 = 2, x_2 = 5, x_3 = 6 \) and \( y = 15 \) then equation (8.6) is satisfied. Thus \((x_1, x_2, x_3, y) = (2, 5, 6, 15)\) is the solution of the Diophantine equation \( x_1^2 + x_2^2 + x_3^2 + 60 = y^2 \).

(ii) If we take \( x_1 = 4, x_2 = 6, x_3 = 7, x_4 = 8 \) and \( y = 15 \) then equation (6) is satisfied. Thus \((x_1, x_2, x_3, y) = (4, 6, 7, 8, 15)\) is the solution of the Diophantine equation \( x_1^2 + x_2^2 + x_3^2 + x_4^2 + 60 = y^2 \).

(B) Diophantine equation \( \sum x_i^3 + a = y^3 \):

(a) For \( a = 0 \) the given Diophantine equation becomes

\[
\sum x_i^3 = y^3.
\]  

If we take \( x_1 = 3, x_2 = 4, x_3 = 5 \) and \( y = 6 \) then equation (7) is satisfied. Thus \((x_1, x_2, x_3, y) = (3, 4, 5, 6)\) is the solution of the Diophantine equation

\[
x_1^3 + x_2^3 + x_3^3 = y^3.
\]

(b) For \( a = 1 \) the given Diophantine equation becomes

\[
\sum x_i^3 + 1 = y^3.
\]  

If we take \( x_1 = 6, x_2 = 8 \) and \( y = 9 \) then equation (8) is satisfied. Thus \((x_1, x_2, y) = (6, 8, 9)\) is the solution of the Diophantine equation

\[
x_1^3 + x_2^3 + 1 = y^3.
\]

(c) For \( a = 2 \) the given Diophantine equation becomes

\[
\sum x_i^3 + 2 = y^3.
\]  

If we take \( x_1 = 5, x_2 = 6 \) and \( y = 7 \) then equation (9) is satisfied. Thus \((x_1, x_2, y) = (5, 6, 7)\) is the solution of the Diophantine equation

\[
x_1^3 + x_2^3 + 2 = y^3.
\]

(d) For \( a = 16 \) the given Diophantine equation becomes

\[
\sum x_i^3 + 16 = y^3.
\]  

If we take \( x_1 = 3, x_2 = 6, x_3 = 7, x_4 = 9 \) and \( y = 11 \) then equation (10) is satisfied. Thus \((x_1, x_2, x_3, x_4, y) = (3, 6, 7, 9, 11)\) is the solution of the Diophantine equation

\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.
\]

(e) For \( a = 17 \) the given Diophantine equation becomes

\[
\sum x_i^3 + 17 = y^3.
\]  

If we take \( x_1 = 3, x_2 = 5, x_3 = 7 \) and \( y = 8 \) then equation (11) is satisfied. Thus \((x_1, x_2, x_3, y) = (3, 5, 7, 8)\) is the solution of the Diophantine equation

\[
x_1^3 + x_2^3 + x_3^3 + 17 = y^3.
\]
\[ x_1^3 + x_2^3 + x_3^3 + 17 = y^3. \]

(f) For \( a = 18 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 18 = y^3. \]

If we take \( x_1 = 2, x_2 = 4, x_3 = 8, x_4 = 9 \) and \( y = 11 \) then equation (12) is satisfied. Thus 
\((x_1, x_2, x_3, x_4, y) = (2, 4, 8, 9, 11)\) is the solution of the Diophantine equation

\[ x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3. \]

(g) For \( a = 20 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 20 = y^3. \]

If we take \( x_1 = 5, x_2 = 7, x_3 = 8 \) and \( y = 10 \) then equation (13) is satisfied. Thus 
\((x_1, x_2, x_3, y) = (5, 7, 8, 10)\) is the solution of the Diophantine equation

\[ x_1^3 + x_2^3 + x_3^3 + 17 = y^3. \]

(h) For \( a = 20 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 20 = y^3. \]

If we take \( x_1 = 2, x_2 = 3, x_3 = 6, x_4 = 9 \) and \( y = 10 \) then equation (14) is satisfied. Thus 
\((x_1, x_2, x_3, x_4, y) = (2, 3, 6, 9, 10)\) is the solution of the Diophantine equation

\[ x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3. \]

(i) For \( a = 30 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 30 = y^3. \]

If we take \( x_1 = 5, x_2 = 6, x_3 = 7 \) and \( y = 9 \) then equation (15) is satisfied. Thus 
\((x_1, x_2, x_3, y) = (5, 6, 7, 9)\) is the solution of the Diophantine equation

\[ x_1^3 + x_2^3 + x_3^3 + 17 = y^3. \]

(j) For \( a = 50 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 50 = y^3. \]

If we take \( x_1 = 1, x_2 = 4, x_3 = 6, x_4 = 10 \) and \( y = 11 \) then equation (16) is satisfied. Thus 
\((x_1, x_2, x_3, x_4, y) = (1, 4, 6, 10, 11)\) is the solution of the Diophantine equation

\[ x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3. \]

(k) For \( a = 55 \) the given Diophantine equation becomes

\[ \sum x_i^3 + 55 = y^3. \]

...(17)
If we take \( x_1 = 3, \ x_2 = 4, \ x_3 = 5, \ x_4 = 9 \) and \( y = 10 \) then equation (17) is satisfied. Thus 
\[
(x_1, x_2, x_3, x_4, y) = (3,4,5,9,10)
\]
is the solution of the Diophantine equation
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.
\]

(i) For \( a = 55 \) the given Diophantine equation becomes
\[
\sum x_i^3 + 55 = y^3.
\]

If we take \( x_1 = 1, \ x_2 = 3, \ x_3 = 4, \ x_4 = 5, \ x_5 = 6, \ x_6 = 8 \) and \( y = 10 \) then equation (18) is satisfied. Thus 
\[
(x_1, x_2, x_3, x_4, x_5, x_6, y) = (1,3,4,5,6,8,10)
\]
is the solution of the Diophantine equation
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + 55 = y^3.
\]

(m) For \( a = 19 \) the given Diophantine equation becomes
\[
\sum x_i^3 + 19 = y^3.
\]

If we take \( x_1 = 2, \ x_2 = 3, \ x_3 = 7, \ x_4 = 11 \) and \( y = 12 \) then equation (19) is satisfied. Thus 
\[
(x_1, x_2, x_3, x_4, y) = (2,3,7,11,12)
\]
is the solution of the Diophantine equation
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + 19 = y^3.
\]

(n) For \( a = 19 \) the given Diophantine equation becomes
\[
\sum x_i^3 + 19 = y^3.
\]

If we take \( x_1 = 5, \ x_2 = 7, \ x_3 = 8, \ x_4 = 9 \) and \( y = 12 \) then equation (20) is satisfied. Thus 
\[
(x_1, x_2, x_3, x_4, y) = (5,7,8,9,12)
\]
is the solution of the Diophantine equation
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + 19 = y^3.
\]

(o) For \( a = 21 \) the given Diophantine equation becomes
\[
\sum x_i^3 + 21 = y^3.
\]

If we take \( x_1 = 2, \ x_2 = 3, \ x_3 = 6, \ x_4 = 10 \) and \( y = 12 \) then equation (22) is satisfied. Thus 
\[
(x_1, x_2, x_3, x_4, y) = (2,3,6,10,12)
\]
is the solution of the Diophantine equation
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + 21 = y^3.
\]

8.3 Concluding Remarks: Here the Diophantine equation \( \sum x_i^2 + a = y^2 \) has been discussed for \( a = 10, 20, 30, 40, 50 \) and 60. Also the Diophantine equation \( \sum x_i^3 + a = y^3 \) has been discussed for \( a = 0, 1, 2, 16, 17, 18, 19, 20, 21, 30, 50 \) and 55. Possible solutions of these Diophantine equations have been obtained. These Diophantine equations can be discussed for other values of \( a \).
References:


