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On Blow-up Time and Rate Of The Numerical Solutions of The Semilinear Heat Equation with Reaction Term

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Abstract

In this paper, we study the numerical blow-up solutions and times of the semilinear heat equation with reaction term. We compute the blow-up growth rate in the numerical solutions of two numerical experiments, depending on the blow-up solutions and times, those have been computed using a finite difference method.

Key words: Semilinear heat equation; blow-up solution; blow-up rate estimate; finite difference.

1. Introduction

In this paper, we study the numerical approximation of solutions, which achieve blow-up in finite time of the semilinear heat equation with reaction term, namely:

$$\left.\begin{array}{lr} u_t = u_{xx} + f(u), & -1 < x < 1, \quad t > 0 \\ u(-1,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & -1 < x < 1 \end{array}\right] \qquad (1)$$

where, f is nonnegative superliear smooth function, and u_0 is positive symmetric function in $(-1,1)$, and nonincreasing in (0,1) vanishes at x=-1, x=1, takes it maximum at $x = 0$, and $u_0'' + f(u) \geq 0$, for $x \in (-\delta, \delta)$, $\delta \leq 1$.

From these properties of initial function, any solution of problem (1) has to be positive symmetric and decreasing in (0,1), and increasing in time for $x \in (-\delta, \delta)$, see [9 ,10].

The problem of semilinear parabolic equation has been introduced in [5,6, 7, 9, 10,12]. For instance, in [5] Friedman and McLeod have studied problem (1), under fairly general assumptions on f (for example f is of power or exponential type). It has been proved that the solutions of this problem blow up in finite time at only a single point, which is x=0, i.e. there exists T > 0, such that:

$$u(0,t) \to \infty, \quad as\ t \to T^-.$$

Moreover, For the special case, where $f(u) = u^p$, $p > 1$, it has been shown that:
for a fixed t and for any $\alpha \geq \frac{2}{p-1}$ the upper point wise estimate takes the following form:

$$u(x,t) \leq \frac{c}{|x|^\alpha}, \qquad x\epsilon[-1,1]/\{0\}, \quad t \in (0,T)$$

On the other hand, it has been shown in [5], that the upper (lower) blow-up rate estimate take the following form

$$\frac{C_1}{(T-t)^{\beta}} \le u(0,t) \le \frac{C_2}{(T-t)^{\beta}} \quad , \; t \in (0,T), \quad \beta = \frac{1}{p-1}, \;\; C_1, C_2 > 0$$

While, where $f(u) = e^u$, it has been shown, that the upper point wise estimate takes the following form:

$$u(x,t) \le \frac{1}{2\alpha}[\log C - m \log(|x|)], \quad x \in [-1,1]/\{0\}, \quad t \in (0,T)$$

where $C > 0, m > 0, \; \alpha \in (0,1/2]$

While, the upper and lower blow-up rate estimate take the following form

$$\log C_1 - \log(T-t) \le u(x,t) \le \log C_2 - \log(T-t), \quad x \in [-1,1], \; t \in (0,T),$$

Where C_1, C_2 are positive constant. For more details about blow-up phenomena, see [10].

In fact, little attention has been devoted to the numerical study for this problem, however, it has been studied numerically by some authors, see [1,2,3,4,8,11].

In [1,2] , it has been considered the semidiscrete and fully discrete problems based on uniform discretizations, but it was mainly concerned with their blow-up times and their convergence to the blow-up time of (1). It has also considered more general nonlinear terms $f(u)$ and assumes that the function f is at least defined on $[0,\infty)$.

In this paper, we use the Euler explicit method, to find the numerical blow-up solutions and blow-up time of problem (1), where f is of power or exponential type, and we shill compute the upper blow-up bound for these problems, using the numerical results.

2. The semidiscrete problem

For J a positive integer, we set $h = 2/J$, and we define the grids:

$$x_0 = -1, \qquad x_J = 1, \qquad \text{and} \quad x_j = x_{j-1} + h \;\; \text{for} \; j = 1,2,\dots,J-1$$

Also, we introduce the time step k> 0 and the discrete time levels

$t_0 = 0, \qquad t_{n+1} = t_n + k = \text{nk}, \quad \text{for n} = 0,1,2,\dots$

We shill denote by U_j^n the approximate value of $u(x_j,t_n)$, for $1 \le j \le J-1$, and $n > 0$, obtained by numerical methods.

We approximate $u_{xx}(x_j,t_n)$ by the standard second order finite difference operator, while $u_t(x_j,t_n)$ is approximated by the forward finite difference operator. Thus, the discrete equation of the semilinear equation in (1), becomes:

$$\frac{U_j^{n+1} - U_j^n}{K} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + f(U_j^n), \quad for \quad 1 \le j \le J-1, n > 0$$

i.e. the discrete problem of problem (1), becomes:

$$\left.\begin{aligned} U_j^{n+1} = (1-2r)U_j^n + r\left(U_{j+1}^n + U_{j-1}^n\right) + kf\left(U_j^n\right), \quad for \quad 1 \le j \le J-1, n > 0,\\ U_0^n = U_J^n = 0,\\ U_j^0 = u(x_j,0) = u_0(x_j), \quad for \quad 0 \le j \le J \end{aligned}\right] \qquad (2)$$

where $r = k/h^2$

It is well known that, $2r \le 1$, the well-known stability condition of the explicit Euler method for the heat equation. Therefore, *h* and *k* must be chosen such that this condition is satisfied. Moreover, the rate of convergence for this method is $O(k + h^2)$.

The next theorem summaries some results, those have been proved in [1], which guarantees that for small *h*, the solution of the discrete problem (2) convergences to the exact solution of (1), for more general nonlinear terms *f* (including power and exponential types).

Theorem 1. Let $U^n = [U_0^n, U_1^n, \dots \dots, U_J^n]$, is the solution of (2). If $0 < k \le \frac{h^2}{2}$, then

1- The solution of (2) is nonnegative, and $U_j^{n+1} \ge U_j^n$, $U_{J-j}^n = U_j^n$ for $0 \le j \le J, n \ge 0$

2- $\|U^n - u^n\| \le C h^2$, $n \ge 0$,

where $u^n = [u_0(x_j), u(x_1, t_n), \dots \dots, u(x_J, t_n)]$.

i.e. $U^n \to u^n$, where $h \to 0$

which means, at each point (x_j, t_n), the numerical solution given by (2) convergences to the exact solution of problem (1).

3. Blow-up in the discrete problem

The solutions of (2) do not exist for all $n \in N$, because they become unbounded for some *n*. We denote $\|U^n\|_\infty = \max_{j=0,\dots,J} |U_j^n|$.

Definition. Let $\{U^n\} = (U_0^n, U_1^n, \dots . U_J^n)$ $n \ge 0$ is the numerical solution of problem (2), we say that $\{U^n\}$ achieves blow-up in the finite time, if there exists $m \in N$ such that

$$\|U^m\|_\infty \to \infty \ .$$

Moreover, $T_J^m = mk$ is called the numerical blow-up time of problem (1).

In fact, the numerical blow-up time depends on the size of spatial grid *h* and also on the choice of time steps *k*.

The next theorem, which has been proved in [1], establishes the convergence of the blow-up time of the approximate discrete problem (2) to the blow-up time of the theoretical solution of (1), for more general types of *f* (including power and exponential types).

Theorem 2. Let $\{U^n\} = (U_0^n, U_1^n, \dots . U_J^n)$ $n \ge 0$ is the solution of (2), such that $0 < k \le \frac{h^2}{2}$, then

1- There exist $m \in N$ such that $\{U^n\}$ achieves blow-up in the finite time T_J^m

2- Let T be the blow-up time of (1), then $T_J^m \to T$, where $J \to \infty$, which means, the numerical blow-up time approaches to the theoretical blow-up time, for h sufficiently small.

4. Numerical Experiments

In this section, we use the discrete formula (2), to study the numerical solutions of (1) for two special cases of the nonlinear term *f*, with initial function $u_0(x) = 5(1 - x^2)$, namely:

$$\left.\begin{array}{ll} u_t = u_{xx} + e^u, & -1 < x < 1, \quad t > 0 \\ u(-1, t) = u(1, t) = 0, & t > 0 \\ u(x, 0) = u_0(x), & -1 < x < 1 \end{array}\right] \qquad \text{(I)}$$

$$\left.\begin{array}{ll} u_t = u_{xx} + u^2, & -1 < x < 1, \quad t > 0 \\ u(-1, t) = u(1, t) = 0, & t > 0 \\ u(x, 0) = u_0(x), & -1 < x < 1 \end{array}\right] \qquad \text{(II)}$$

It is clear that u_0 satisfies all our assumptions, and takes its maximum value at the point $x = 0$, therefore according to the known blow-up results for the problem (1) (see [5]), the blow-up in equation (4), (5) occurs only at a single point, which is $x = 0$.

For each problem of **I** and **II**, our aim is to use the discrete equation (2) to compute numerically, the blow-up time and solutions, then we will use these results to compute the upper blow-up bounds.

In fact, the blow-up time will be taken experimentally, at the first time that $\|U^n\|_\infty \ge 10^6$

For converging, we will choose, $k = h^2/2$, and we will get a symmetric numerical solution which takes its maximum at $x_{J/2} = 0$, with respect to the meshes *J*=10,20,30,…100

In order to compute the numerical blow-up bound, depending on the theoretical forms of the blow-up rate estimates of problems (1) and (II), at each fixed value of *J*, we need to compute the constant of growth rate C_J, using the relation $C_J = \max_{0 \le n \le m-1} C(n)$,

where $C(n) = U^n_{J/2} + \log\left(T_J^m - t_n\right)$, for problem (I)

$C(n) = U^n_{J/2}\left(T_J^m - t_n\right)$, for problem (II)

The problems were solved by using Matlab programming. In the next tables (1) and (2), we show the numerical blow-up times and the numerical blow-up constant of growth rate C_J, for problem (I) and (II) respectively, with respect to the meshes *J*=10,20,30,…100. While in table (3) and (4) we present the iterative errors those can be got form using the error form $E_J = |T^m_{J+10} - T_J^m|$, where *m* is referred to the number of iteration, when numerical blow-up occurs.

Table (1), Numerical blow-up times and the constants of growth rates for problem (I)

J	k	m	$T = mk$	C
10	0.0200	3	0.0600	50.4
20	0.0050	5	0.0250	1.9582e+03
30	0.0022	7	0.0156	316.2538
40	0.0013	10	0.0125	5.9655e+03
50	8.0000e-04	13	0.0104	56.0644
60	5.5556e-04	17	0.0094	29.1636
70	4.0816e-04	22	0.0090	65.6771
80	3.1250e-04	28	0.0088	1.3164e+04
90	2.4691e-04	34	0.0084	356.7500
100	2.0000e-04	41	0.0082	590.5392

Table (2), Numerical blow-up times and the constants of growth rates for problem (II)

J	k	m	$T = mk$	C
10	0.0200	22	0.4400	2.7322e+03
20	0.0050	70	0.3500	187.3883
30	0.0022	150	0.3333	1.1890e+03
40	0.0013	260	0.3250	50.0727
50	8.0000e-04	403	0.3224	756.4616
60	5.5556e-04	576	0.3200	62.6488
70	4.0816e-04	781	0.3188	42.8687
80	3.1250e-04	1018	0.3181	138.4165
90	2.4691e-04	1286	0.3175	101.0971
100	2.0000e-04	1585	0.3170	21.3472

Table(3), Errors in the numerical blow-up times, for problem (I)

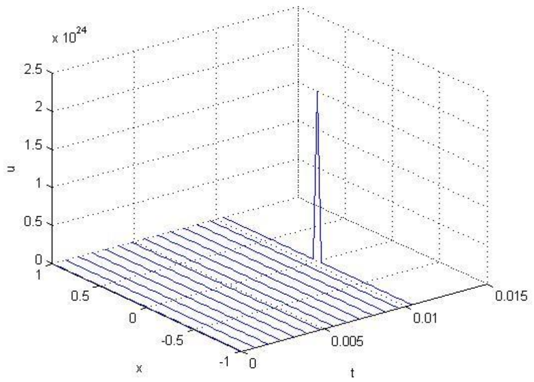
J	$E_J = \|T^m_{J+10} - T_J^m\|$,	J	$E_J = \|T^m_{J+10} - T_J^m\|$,
10	0.035	60	0.0004
20	0.0094	70	0.0002
30	0.0031	80	0.0004
40	0.0021	90	0.0002
50	0.001	100	………

Table(4), Errors in the numerical blow-up times, for problem (II)

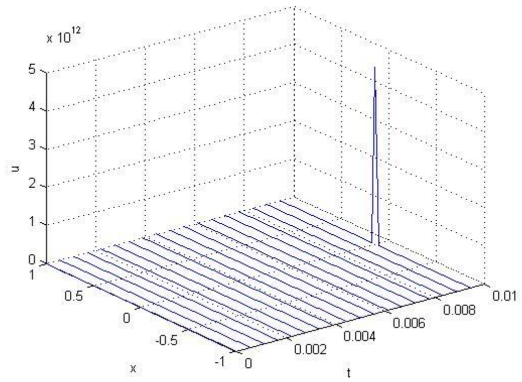
J	$E_J = \|T_{J+10}^m - T_J^m\|,$	J	$E_J = \|T_{J+10}^m - T_J^m\|,$
10	0.09	60	0.0012
20	0.0167	70	0.0007
30	0.0083	80	0.0006
40	0.0026	90	0.0005
50	0.0024	100	……..

The next figures show the evolutions in time, of the numerical blow-up solutions of problem (I) and (II), with respect to some different values of J and $0 \le n \le m$.

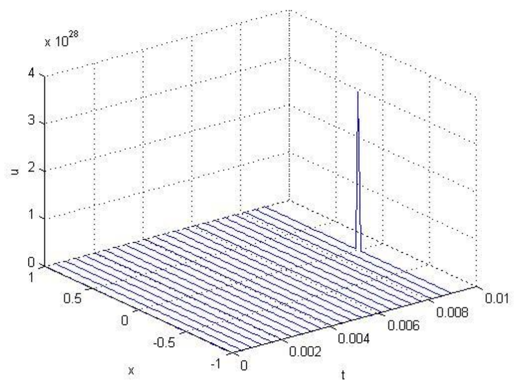
Figures (1), Numerical blow-up solutions of problem (I), with respect to J=50, 60,70,80,90,100



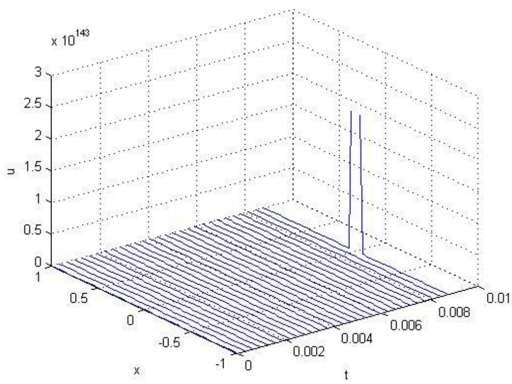
J=50



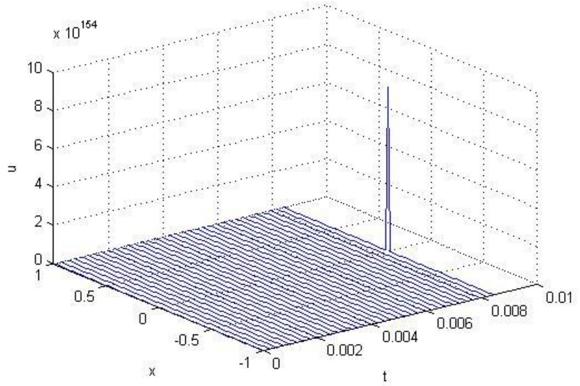
J=60



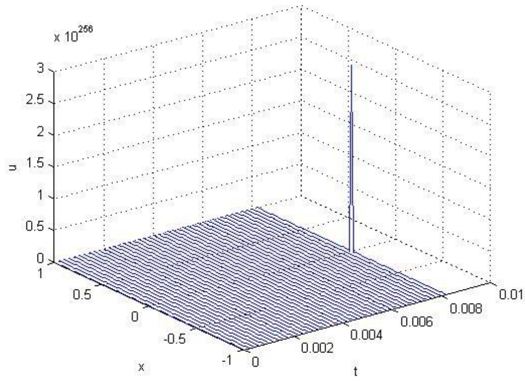
J=70



J=80

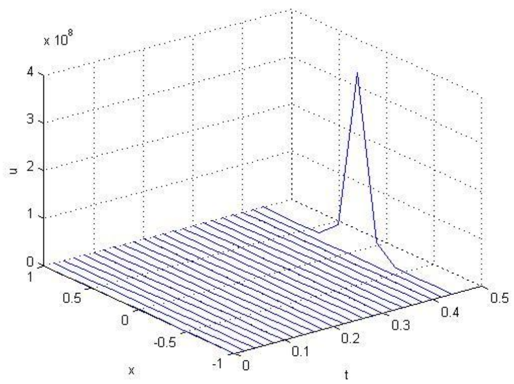


J=90

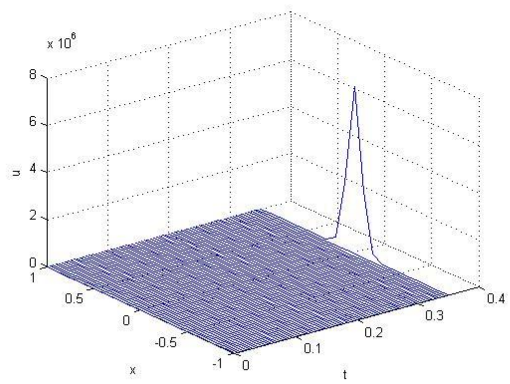


J=100

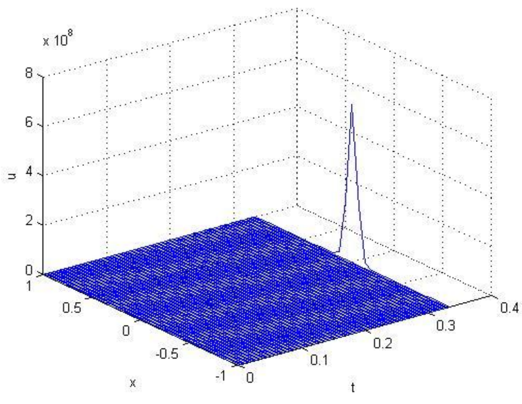
Figures (2), Numerical blow-up solutions of problem (II), with respect to J=10, 20, 30



J=10



J=20



J=30

While, the next two figures, show the declining of errors, where J is increasing

Figure (3), Errors with respect to J, for problem (I)

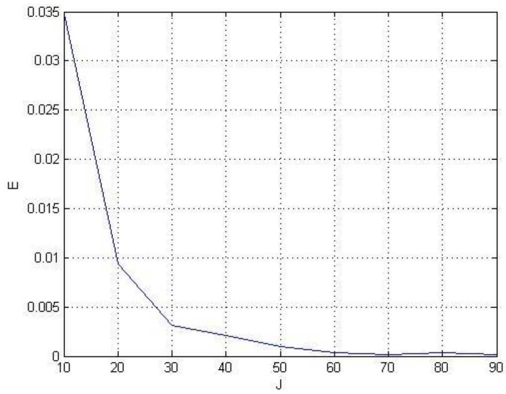
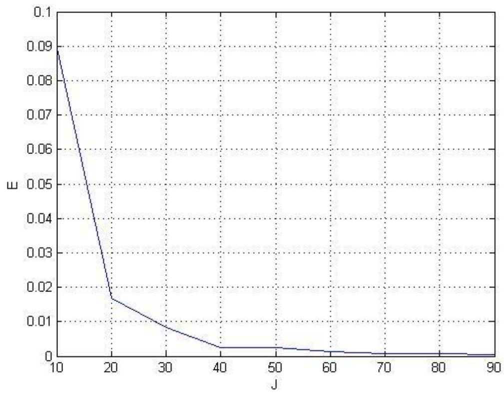


Figure (4), Errors with respect to J, for problem (II)



5. Conclusions

In this section, we point out some conclusions that can be noted from our numerical results (Table 1,2, 3 &4 and Figures 1,2,3 & 4), as follows:

- For each value of J, depending on the value of C_J ,we can compute the numerical upper blow-up rate estimate as follows:

$$U_{J/2}^{n} \leq C_J - \log\left(T_J^{m} - t_n\right), \quad \text{for problem (I)}$$

$$U_{J/2}^{n} \leq \frac{C_J}{\left(T_J^{m} - t_n\right)}, \quad \text{for problem (II)}$$

  Where T_J^{m} is the numerical blow-up time, $0 \leq n \leq m-1$.

- For a fixed value to J, we have found that the corresponding numerical blow-up time is larger than the numerical blow-up time, with respect to J+10, which means: decreasing *h* and *k* leads to decreasing the numerical blow-up time.

- The tables of errors 3,4, in the computed blow-up times, show that, for a fixed value of J, we have almost that: $E_{J+10} < E_J$, which means: decreasing *h* and *k* leads to decreasing the iterative errors.

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