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Moufang Loops of Odd Order $p_1^3p_2^3\cdots p_n^3$.

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Abstract

In [10], all Moufang loops of order p^3q^3 where p and q are odd primes with $p<q$, were shown to be associative if and only if $q \not\equiv 1 (mod\ p)$. In this paper, we generalize this result to Moufang loops of order $p_1^3p_2^3\cdots p_n^3$ for distinct odd primes $p_1, p_2, \cdots, p_n$ with $p_i \not\equiv 1(mod\ p_j)$ for every $i, j \in \{1,\ldots,n\}$.

Keywords: Loop; maximal subloop; order; nonassociative.

1. Introduction

A loop $\langle L,\cdot\rangle$ is said to be a Moufang loop if for any $u,v,w \in L$ the identity $(u\cdot v)\cdot(z\cdot u) = (u\cdot(v\cdot w))\cdot u$ is satisfied. It has been proven that Moufang loops can be classified and resolved once the order of the given Moufang loop is known. Moufang loops of even order have been successfully resolved and the odd order is still being resolved.

Verifying the associativity or nonassociativity of Moufang loops of odd order $p_1^3p_2^3\cdots p_n^3$ was put forward by Leong F. and Rajah A. in [3]. Over a period of several time other results were obtained and culminated in the resolution of Moufang loops of odd order p^3q^3 and pq^4 where p and q are odd primes with $p<q$ and $q \not\equiv 1(mod\ p)$ as seen in [10] and [11] respectively. As a result of the aforementioned the possibility of resolving all Moufang loops of odd order $p_1^3p_2^3\cdots p_n^3$ now became the next focus and has been resolved as presented in this paper.

2. Definitions and Notations

The following definitions are quite standard. One can refer to [1], [2], for further details.

1. A loop $\langle L,\cdot\rangle$, is a binary system that satisfies the following two conditions: (i) specification of any two of the elements u, v, w in the equation $u\cdot v = w$ uniquely determines the third element, and (ii) the binary system contains an identity element (we denote it as 1).

2. A Moufang loop is a loop $\langle L,\cdot\rangle$ such that $(u\cdot v)\cdot(z\cdot u) = (u\cdot(v\cdot w))\cdot u$ for any $u,v,w \in L$. (From now on, for the sake of brevity, we shall simply refer to the loop $\langle L,\cdot\rangle$ as the loop L. Also, we shall write $(u\cdot v)\cdot w$ simply as $uv\cdot w$, $(u\cdot(v\cdot w))\cdot u$ as $(u\cdot vw)u$, e.t.c.)

3. The associator subloop of L is denoted as $L_a = (L, L, L) = \langle (l_1, l_2, l_3) \mid l_i \in L \rangle$. In a Moufang Loop, L_a is the subloop generated by all the associators $(u, v, w) \in L$ such that $(u, v, w) = (u \cdot vw)^{-1}(uv \cdot w)$. It is obvious that L is associative if and only if $L_a = \{1\}$.

4. $I(L) = \langle R(u, v), L(u, v), T(u) | u, v \in L \rangle$ is called the inner mapping group of L, where

$$wR(u, v) = (wu \cdot v)(uv)^{-1}$$

$$wL(u, v) = (vu)^{-1}(v \cdot uw)$$

$$wT(u) = u^{-1} \cdot wu$$

5. The commutator subloop of L, denoted L_c, is the subloop generated by all commutators $[u, v]$ in L, where $uv = vu[u, v]$.

6. The subloop generated by all $n \in L$ such that $(n, u, v) = (u, n, v) = (u, v, n) = 1$ for any $u, v \in L$ is called the nucleus of L. It is denoted as $N(L)$ or simply as N.

7. Suppose R is a subloop of L. Then

$$C_L(R) = \{u \in L | ur = ru \text{ for all } r \in R\}.$$

8. Let M be a subloop of L and π a set of primes.

(a) M is a normal subloop of L, denoted $M \lhd L$, if $M\theta = M$ for all $\theta \in I(L)$.

(b) A positive integer n is a π-number if every prime divisor of n lies in π.

(c) For each positive integer n, we let n_π be the largest π-number that divides n.

(d) M is a π-loop if the order of every element of M is a π-number.

(e) M is a Hall π-subloop of L if $|M| = |L|_\pi$.

(f) M is a Sylow p-subloop of L if M is a Hall π-subloop of L and π contains only a single prime p.

9. Assume M is a normal subloop of L.

(a) M is a proper normal subloop of L if $M \neq L$.

(b) L/M is a proper quotient loop of L if $M \neq \{1\}$.

10. Assume M is a normal subloop of L.

(a) M is a minimal normal subloop of L if M is non-trivial and contains no proper nontrivial subloop which is normal in L. In other words, if there exists $H \lhd L$ with $\{1\} < H < M$, then $H = \{1\}$ or M.

(b) M is a maximal normal subloop of L if M is not a proper subloop of every other proper normal subloop of L. In other words, if there exists $H \triangleleft L$ such that $M < H$, then $M = H$ or $H = L$.

11. (m,n) is defined as the greatest common divisor of the integers m and n.

3. Basic properties and known results

Let L be a Moufang loop.

Lemma 3.1. L is diassociative, that is, $\langle u, v \rangle$ a group for any $u, v \in L$. Moreover, if $(u, v, w) = 1$ for some $u, v, w \in L$, then $\langle u, v, w \rangle$ is a group. [10, Moufang's Theorem, p.1360]

Lemma 3.2. $N = N(L)$ is a normal subloop of L [1, Theorem 2.1, p.114]. Clearly N, is a group by its definition.

Lemma 3.3. Suppose $K \triangleleft L$. Then

(a) L/K is a group implies $L_a \subset K$.

(b) L/K is commutative implies $L_c \subset K$.

[6, Lemma 1, p.563]

Note that the properties above hold for all Moufang loops in general. However, the following properties hold only for finite Moufang loops L.

Lemma 3.4. Suppose K is a subloop of L. Then $|K|$ divides $|L|$. [10, Lagrange's theorem, p.1360]

Lemma 3.5. Suppose K is a subloop of L, and π is a set of primes. Then

a. L is solvable. [2, Theorem 16, p.413]

b. If K is a minimal normal subloop of L, then K is an elementary abelian group and

$(K, K, L) = \langle (k_1, k_2, l) \mid k_i \in K, l \in L \rangle = \{1\}$. [2, Theorem 7, p.402]

c. K is a normal subloop of $L, (K, K, L) = \{1\}$ and $(|K|, |L/K|) = 1$ implies $K \subset N$. [2, Theorem 10, p.405]

d. L contains a Hall π -subloop. [2, Theorem 12, p.409]

Lemma 3.6. L is a group if

a. $|L| = p^{\alpha} q_1^{\beta_1} q_2^{\beta_2} \cdots q_n^{\beta_n}$, where p and $q_1 \cdots q_n$ are primes with $p < q_1 < \cdots < q_n$, $\alpha \le 3$ and $\beta_i \le 2$. [4 , Theorem 1, p.482]

b. $|L| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} q^3$, where $p_1, p_2, \cdots, p_n$ and q are distinct odd primes with $q \not\equiv 1 (mod\ p_i)$ and $1 \le \alpha_i \le 2$. [9, Theorem 4.2, p.970]

c. $|L| = p^3q^3$, where p and q are odd primes with $p < q$, and $q \not\equiv 1(mod\ p)$. [10, Theorem 4.6, p.1364] .

Lemma 3.7. Suppose $|L|$ is odd and every proper subloop of L is a group. If there exists a minimal normal Sylow subloop in L, then L is a group. [10, Lemma 3.15a, p.1361]

Lemma 3.8. If there exist H, K in L such that $H \lhd K \lhd L$ and $(|H|, |K/H|) = 1$. Then $H \lhd L$. [10, Lemma 3.10, p.1360]

Lemma 3.9. Let $|L| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} q^{\beta}$, where $1 \le \beta \le 2$ and each p_i is a prime such that

$p_i < q$. Suppose:

i. every proper subloop of L is a group, and

ii. there exists a Sylow q-subloop normal in L.

Then L is a group. [3, Lemma 3, p.879]

Lemma 3.10. Let L be of odd order, K, a minimal normal subloop of L such that $K \subset L$, and Q, a Hall subloop of L. Suppose all proper subloops and proper quotient loops of L are groups, $(|K|, |Q|) = 1$ and $Q \lhd KQ$. Then L is a group. [6, Lemma 3, p.564]

Lemma 3.11. Let L be of odd order such that every proper subloop and proper quotient loop of L is a group. Suppose Q is a Hall subloop of L such that $(|L_a|, |Q|) = 1$, and $Q \lhd L_a Q$. Then L is a group. [6, Lemma 3, p.564]

Lemma 3.12. Let L be nonassociative and of odd order such that all proper quotient loops of L are groups. Then:

a. L_a is a minimal normal subloop of L; and is an elementary abelian group. [10, Lemma 3.16, p.1361]

b. if M is a maximal normal subloop of L, then L_a and L_c lie in M. Moreover, $L = M\langle x \rangle$ for any $x \in L \setminus M$. [7, Lemma 1(b), p.478]

Lemma 3.13. Suppose K is a subloop of $C_L(L_a)$ and $(|K|, |L_a|) = 1$. Then $K \subset N$. [10, Lemma 3.9, p.1360]

Lemma 3.14. Suppose

a. $|L| = p^{\alpha} m$ where p is a prime, $(p, m) = (p-1, p^{\alpha} m) = 1$ and L has an element of order p^{α}. Then there exists a (Sylow p-)subloop P of order p^{α} and a normal subloop M of order m in L such that $L = PM$.

b. $|L| = p^2 m$ where p is the smallest prime dividing $|L|$ and $(p, m) = 1$. Then there exists a subloop P of order p^2 and a normal subloop M of order m in L such that $L = PM$.

[10, Lemma 3.19, p.1361]

Lemma 3.15. Let L be of odd order and K a normal subloop of L. Suppose $K \subset N$. Then there exists a homomorphism from L to $Aut(K)$ with $C_L(K)$ as the kernel. Thus $C_L(K) \lhd L$ and $|L/C_L(K)|$ divides $|Aut(K)|$. [10, Lemma 3.12, p.1360]

Lemma 3.16. Let L be of odd order and K a normal Hall subloop of L. Suppose $K = \langle x \rangle L_a$ for some $x \in K \setminus L_a$ and $L_a \subset N$. Then $K \subset N$. [11, Lemma 3.6, p.427]

Lemma 3.17. Let L be nonassociative and of odd order, and let M be a maximal normal subloop of L. Suppose all proper subloops and proper quotient loops of L are groups. Then

a. L_a is a Sylow subloop of N, then $L_a = N$. [10, Lemma 3.17, p.1361]

b. L_a is cyclic, then $L_a \subset N$. [10, Lemma 3.18a, p.1361]

c. $(k,w,l)=1$ for all $k \in L_a, w \in M, l \in L$, then $L_a \subset N$. [10, Lemma 3.18b, p.1361]

d. $(k,w,l) \neq 1$ for some $k \in L_a, w \in M, l \in L$, then L_a contains a proper nontrivial subloop which is normal in M. [10, Lemma 3.18c, p.1361]

Lemma 3.18. Suppose $|L|$ is odd and every proper subloop of L is a group. If N contains a Hall subloop of L, then L is a group. [10, Lemma 3.15b, p.1361]

Lemma 3.19. Let L be of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} q$, where $p_1, p_2, \cdots, p_n$ and q are odd primes with $p_1 < p_2 < \cdots < p_n < q$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{Z}^+$, $q \not\equiv 1 (mod\ p_i)$ for all i. Then there exists a normal subloop of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ in L. [10, Lemma 4.1, p.1362]

Lemma 3.20. Let L be nonassociative and of odd order, and let M be an associative maximal normal subloop of L and P be a subloop of L. Suppose $L_a \subset N$ and $M = L_a P$. Then for any $x \in L \setminus M$, there exist some $r, s \in P$ such that $(x,r,s) \neq 1$. [10, Corollary 4.3, p.1362]

Lemma 3.21. Let L be of odd order and M be an associative maximal normal subloop of L. Suppose $L_a \subset N$. Then g commutes with (x,g,h) for any $x \in L \setminus M$ and $g, h \in M$. [10, Lemma 4.4, p.1363]

Lemma 3.22. Suppose p and q are distinct odd primes. Then there exists a nonassociative Moufang loop of order pq^3 if and only if $q \equiv 1 (mod\ p)$. [5, Theorem 1, p.78 and Theorem 2, p.86]

Lemma 3.23. If q is a prime, then the congruence $\mu^n \equiv 1 (mod\ q)$ has $(n, q-1)$ solutions for μ. [7, Theorem 2.27, p.54]

Lemma 3.24. Let G be a group and $r,s,t \in G$ with $[r,t]=[s,t]=1$. Suppose $r^{-1}sr = s^{\alpha}t^{\beta}$, for some $\alpha, \beta \in \mathbb{Z}^+$. Then $r^{-n}sr^n = s^{\alpha^n} t^{\beta(\alpha^0 + \alpha^1 + \cdots + \alpha^{n-1})}$ for all $n \in \mathbb{Z}^+$. [10, Lemma 4.5, p.1363]

4. Main result

Lemma 4.1 *Suppose* $|L| = p^{\alpha}m$ *where* p *is the smallest prime dividing* $|L|$ *with* $(p,m)=1$ *and* $\alpha \in \{1,2\}$. *Then there exists a subloop* M *of order* m *normal in* L.

Proof:

Suppose $\alpha = 1$, i.e., $|L| = pm$. Since p is the smallest prime dividing L, $(p-1, pm) = 1$. By Lemma (3.5)(d) there exists an element of order p in L. Then the result follows from Lemma (3.14)(a). However, if $\alpha = 2$, i.e., $|L| = p^2 m$. Then the result follows from Lemma (3.14)(b).

Lemma 4.2 *Let* L *be a Moufang loop of odd order* mq^2 *where* q *is a prime with* $(m,q)=1$ *and* $q \not\equiv 1 (mod\ p_i)$ *for every prime divisor* p_i *of* m. *Suppose there exists a normal subloop of order* q *in* L. *Then there exists a subloop of order* m *normal in* L.

Proof:

Let S be the normal subloop of order q. So $|L/S| = mq$ and by Lemma (3.19), there exists $T/S \lhd L/S$ such that $|T/S| = m$, so $|T| = mq$. Again, there exists $M \lhd T$ such that $|M| = m$ by Lemma (3.19). Since M is a normal Hall subloop in T, $M \lhd L$ by Lemma (3.8).

Lemma 4.3 *Let* L *be a nonassociative Moufang loop of order* $p^{\alpha}m$ *where* p *is a prime and* $(p,m)=1$, *such that all proper subloops and proper quotient loops of* L *are groups. Suppose* $|L_a| = p^{\alpha-1}$ *and* $L_a \subset N$. *Then* $L_a = N = C_L(N)$.

Proof:

Since $L_a \subset N$, by Lemma (3.4) $|L_a| = p^{\alpha-1}$ is a divisor of $|N|$. Also N cannot contain a Hall subloop of L by Lemma (3.18). So p^{α} cannot divide $|N|$. Thus, L_a is a Sylow subloop of N. So by Lemma (3.17)(a), $L_a = N$. Therefore $|N| = p^{\alpha-1}$.

Now L_a is abelian by Lemma (3.5)(b). Then by it's definition, $C_L(N) = C_L(L_a)$ contains L_a. So

$$|L_a| = p^{\alpha-1} \text{ is a divisor of } |C_L(N)|. \tag{3.1}$$

Let q be a prime divisor of m. Then $(p,m) = 1 (p,q) = 1$.

Assume $q \| C_L(N)|$. Then, by Sylow's theorem there exists Q a subloop of order q in $C_L(N)$. So $(|L_a|,|Q|) = 1$ and $Q \subset N$ by Lemma (3.13).

This is a contradiction since $q \nmid |N|$. Therefore, $q \nmid |C_L(N)|$.

So by (3.1), $|C_L(N)| = p^{\alpha-1}$ or p^{α}.

Assume $|C_L(N)| = p^{\alpha}$. Then there exists $x \in C_L(N) \setminus L_a$. So $\langle L_a, x\rangle = C_L(N)$. Since $L_a \lhd L$, $\langle L_a, x\rangle = \langle x\rangle L_a$. Then by Lemma (3.15), $C_L(N)$ is a normal Hall subloop of L, since $N \lhd L$ by Lemma (3.2). So by Lemma (3.16) we have that $C_L(N) \lhd N$, which is impossible since $|C_L(N)| = p^{\alpha}$ and $|N| = p^{\alpha-1}$. So $|C_L(N)| \neq p^{\alpha}$. Therefore $|C_L(N)| = p^{\alpha-1}$, i.e., $C_L(N) = C_L(L_a) = L_a = N$.

Lemma 4.4 *Let L be a nonassociative Moufang loop of order $p_1^{\alpha_1}\cdots p_n^{\alpha_n}q^3, n, \alpha_i \in Z^+$ where $p_1, p_2, \cdots, p_n, q$ are distinct odd primes with $p_i < q$, $q \not\equiv 1(mod\ p_i)$ and $p_j \not\equiv 1(mod\ p_i)$ such that all proper subloops and proper quotient loops of L are groups. Suppose M is a maximal normal subloop of order $p_1^{\alpha_1}\cdots p_n^{\alpha_n}q^2$ in L, and for all $k \in L_a$, $w \in M$ and $l \in L$, $(k, w, l) = 1$. Then $|L_a| \neq q^2$.*

Proof:

Suppose this is not the case, i.e., $|L_a| = q^2$. Since for all $k \in L_a, w \in M$ and $l \in L$, it follows that $L_a \subset N$. Thus

$$C_L(N) = N \tag{3.2}$$

by Lemma (3.3). Now $|L/M| = q$. So L/M is an abelian group. Hence

$$L_a, L_c \subset M \tag{3.3}$$

by Lemma (3.3). M contains a Hall subloop H of order

$$M \text{ contains a Hall subloop } H \text{ of order } p_1^{\alpha_1}\cdots p_n^{\alpha_n} \tag{3.4}$$

by Lemma (3.5)(d). Also, since $L_a \lhd L$, by (3.3), $L_a \lhd M$. Thus $L_a H < M$ where $|L_a H| = \dfrac{|La||H|}{|L_a \cap H|} = q^2 p_1^{\alpha_1}\cdots p_n^{\alpha_n} = |M|$. Hence $M = L_a H$. By Lemma (3.20), we get that:

$$\text{Whenever there exists some } r, s \in H \text{ for any } x \in L \setminus M,\ (x, r, s) \neq 1. \tag{3.5}$$

Now $|L_a| = q^2, i.e, L_a = N$ and by Lemma (3.12)(a) $L_a = C_q \times C_q$. (3.6)

Write $t = (x, r, s)$. By (3.6),

$$L_a = \langle t\rangle \times \langle u\rangle \tag{3.7}$$

for some $u \in L_a \setminus \langle t\rangle$. So

$$[r, t] = 1, \tag{3.8}$$

by Lemma (3.21) since $H \subset M$ by (3.4).

The fact that $L_a \lhd L$ and $u \in L_a$ means $r^{-1}ur \in L_a$. So by (3.7) we can express

$$r^{-1}ur = u^{\lambda}t^{\eta} \tag{3.9}$$

for some $\lambda, \eta \in Z^{+}$. Now $(u,t,r)=1$ since $t \in L_a = N$. Thus the elements u,t and r associate. So $u = r^{-|r|}ur^{|r|} = u^{\lambda^{|r|}}t^{\eta(\lambda^0+\lambda^1+\cdots+\lambda^{|r|-1})}$ by Lemma (3.24) since $|r| \in Z^{+}$ as $r \neq 1$ by (3.5). Then

$$u^{1-\lambda^{|r|}} = t^{\eta(\lambda^0+\lambda^1+\cdots+\lambda^{|r|-1})} = 1. \tag{3.10}$$

$u^{1-\lambda^{|r|}} = 1$ since $\langle t \rangle \cap \langle u \rangle = \{1\}$. So $q_n \mid (1-\lambda^{|r|})\lambda^{|r|} \equiv 1 (mod\ q_n)$.

Now since $r \in H$, $|r| = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_n^{\gamma_n}$ where $0 \leq \gamma_i \leq \alpha_i$. This implies $(|r|, q-1) = p_1^{\delta_1} p_2^{\delta_2} \cdots p_n^{\delta_n}$ where $0 \leq \delta_i \leq \gamma_i$. Assume $\delta_i > 0$ for some i. Then $p_i \mid (q-1)$. Then we get $q \equiv 1 (mod\ p_i)$ which is a contradiction. Hence $\delta_i = 0$ for all i. So

$$(|r|, q-1) = 1. \tag{3.11}$$

Now $\lambda = 1$ is a solution for the congruence $\lambda^{|r|} \equiv 1 (mod\ q_n)$ and is the only possible solution for this congruence by (3.10) and Lemma (3.23).

Now $\lambda^0 + \lambda^1 + \cdots + \lambda^{|r|-1} = \underbrace{(1+1+\cdots+1)}_{|r| \text{ times}} = |r|$. So $t^{\eta(\lambda^0+\lambda^1+\cdots+\lambda^{|r|-1})} = t^{\eta|r|} = 1$ by (3.10). Thus $|t| = q_n$ divides $\eta|r|$. Since q is not a factor of $|r|$, $q \mid \eta$. This means $t^{\eta} = 1$. Hence $r^{-1}ur = u$ by (3.9), i.e, $[r,u] = 1$. So r commutes with both generators of L_a by (3.7) and (3.8). Therefore, $r \in C_L(L_a) = C_L(N) = N$ by (3.2). This is a contradiction since $|r| \nmid |N|$. So the assumption is false. Hence $|L_a| \neq q^2$.

Theorem 4.5 *Let* $p_1, p_2, \cdots, p_n$ *be distinct odd primes. Then all Moufang loops of order* $p_1^3 p_2^3 \cdots p_n^3$ *are associative if and only if* $p_i \not\equiv 1 (mod\ p_j)$ *for every* $i, j \in \{1, \ldots, n\}$.

Proof:

Now, let $p_1, p_2, \cdots, p_n$ be distinct odd primes.

By [10], the result is true when $n = 1$ or 2. So we shall assume that

$$n \geq 3. \tag{3.12}$$

Suppose $p_i \equiv 1 (mod\ p_j)$ for some $i, j \in \{1, \ldots, n\}$, Then by [5], there exists a nonassociative Moufang loop of order $p_j p_i^3$. So by using a direct product of this loop with any group (for example, the cyclic group) of order $p_1^3 p_2^3 \cdots p_n^3 / p_j p_i^3$, we would obtain a nonassociative Moufang loop of order $p_1^3 p_2^3 \cdots p_n^3$. So all Moufang loops of order $p_1^3 p_2^3 \cdots p_n^3$ would be associative only if $p_i \not\equiv 1 (mod\ p_j)$ for every $i, j \in \{1, \ldots, n\}$.

Now, suppose $p_i \not\equiv 1(mod\ p_j)$ for every $i, j \in \{1,\ldots,n\}$ and assume that the theorem is false. Then there must exist a (minimally)

nonassociative Moufang L of odd order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ (3.12)

with each $\alpha_i \le 3$, such that

if m is a proper divisor of $|L|$, then all Moufang loops of order m are associative. (3.13)

Then by Lemma (3.6)(a), (b) and (c),

$\alpha_k = 3$ for at least one $k \in \{2,\ldots,n-1\}$; or $\alpha_1 = \alpha_n = 3$. (3.14)

Also by (3.13) and Lagrange's theorem, every proper subloop and proper quotient loop of L is associative.

By Lemma (3.5)(a), L is solvable.

Then L_a, by Lemma (3.12), is a minimal normal subloop of L and by Lemma (3.5)(b) is also an elementary abelian group.

So, $|L_a| = p_i^{\gamma_i}$, for some $1 \le \gamma_i \le \alpha_i \le 3$. By Lemma (3. 7), L_a is not a Sylow subloop of L. So $\gamma_i \ne \alpha_i$. Hence

$$1 \le \gamma_i < \alpha_i \le 3. \tag{3.15}$$

If $\alpha_i = 1$, no value of γ_i would satisfy (3.15). So, we have two possible values for α_i, i.e., $\alpha_i = 2$ (with $\gamma_i = 1$) or $\alpha_i = 3$ (with $\gamma_i = 1$ or 2). Hence

$$|L_a| = p_i \text{ (with } \alpha_i = 2 \text{ or } 3) \text{ or } |L_a| = p_i^2 \text{ (with } \alpha_i = 3). \tag{3.16}$$

(Note: i is now a fixed value in $\{1,2,\cdots,n\}$.)

Case 1: Suppose $i \ne n$

There exists P_n a Sylow p_n-subloop of order $p_n^{\alpha_n}$ in L by Lemma (3.5)(d).

Now $L_a \lhd L$, so $L_a P_n < L$ and $|L_a P_n| = \dfrac{|L_a||P_n|}{|L_a \cap P_n|} = p_i p_n^{\alpha_n}$ or $p_i^2 p_n^{\alpha_n}$ by (3.16). Furthermore, since P_n is a Sylow p_n-subloop of $L_a P_n$, $P_n \lhd L_a P_n$ by Lemma (4.1). Also, since $(|P_n|, |L_a|) = (p_n^{\alpha_n}, p_i^{\gamma_i}) = 1$, L is associative by Lemma (3.10). This negates the assumption (3.12).

Case 2: Suppose $i = n$

Then $|L_a| = p_n$ or p_n^2.

Case 2.1: Suppose $|L_a| = p_n$

Then there exists P a Hall subloop of order $p_1^{\alpha_1}$ in L by Lemma (3.5)(d). Now $L_a \lhd L$ so $L_aP < L$ and $|L_aP| = \frac{|L_a||P|}{|L_a \cap P|} = p_n p_1^{\alpha_1}$ since $p_n \neq p_1$, by (3.14). Furthermore, since P is a Sylow p_1-subloop of L_aP, $P \lhd L_aP$ by Lemma (3.19). Also, since $(|P|,|L_a|) = (p_1^{\alpha_1}, p_n) = 1$, L is associative by Lemma (3.10). This also negates the assumption (3.12).

Case 2.2: Suppose $|L_a| = p_n^2$

Then $\alpha_n = 3$ by (3.15). Now by Lemma (3.12) $L_a \lhd L$. Therefore $|L/L_a| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} p_n$. Furthermore, there exists a subloop $M/L_a \lhd L/L_a$ such that $|M/L_a| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}}$ by Lemma (3.19) and (3.14). So $M \lhd L$ and $|M| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} p_n^2$. Hence M is a maximal normal subloop of L.

Now since $|L_a| = p_n^2$, by Lemma (4.4) for some $k \in L_a, w \in M$ and $l \in L$, $(k,w,l) \neq 1$.

L_a, by Lemma (3.17)(d) contains S a proper non trivial subloop normal in M. Thus, $|S| = p_n$. So there exists $R \lhd M$ such that $|R| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}}$ by Lemma (4.2). Since R is a normal Hall subloop in M, $R \lhd L$ by Lemma (3.8). Since L/R is associative, $L_a \subset R$ by Lemma (3.3)(a). Then $|L_a|$ is a divisor of $|R|$ by Lemma (3.4), which is a contradiction since $|L_a| = p_n^2$ and $|R| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}}$.

Since the assumption (3.12) leads to a contradiction in every possible case, it follows that L is associative.

Conclusion

Study on Moufang loops of odd order in this direction is still far from completion. The main theorem in this paper and the result in [5] raised the next open case as stated below:

Does there exist a nonassociative Moufang loop of order p^2q^4, where p and q are odd primes with $p < q$ and $q \not\equiv 1 (mod\ p)$?

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